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COUPLING THERMAL TRANSFER WITH VISCOELASTIC DEFORMATION USING GENERALIZED KELVIN CHAINS

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Abstract. Coupling thermal transfer with deformation caused by predominantly mechanical loads belongs to important tasks of engineering computational mechanics. Simplifying assumptions working with purely elastic deformation cannot describe real energy dissipation, thus some more general computational models are needed, as viscoelastic or elastoplastic ones, with potential incorporation of microscopic and/or macroscopic damage, with the possibility of both their formal mathematical verification and practical validation based on laboratory experiments and observations in situ. All resulting models are then compromises between i) the complexity of considered physical processes, ii) some reasonable setting of material parameters, based on sufficiently simple experiments, iii) some formulation of a transparent mathematical model, using available results from (nearly) linear functional analysis, iv) the design of robust and effective computational algorithms. In this short paper we shall pay attention namely to so-called generalized viscoelastic Kelvin chains, consisting of parallel or serial viscous and elastic terms, with a three-component standard linear solid as a quasi-static model problem. Such detailed analysis will be followed by brief comments to several non-linear generalizations of this approach, including still unclosed problems, as the challenge for future research.

Keywords: Thermal transfer, viscoelastic deformation, computational modelling, method of discretization in time, finite element method.

1. Introduction

Reliable computational modelling of strains and stresses in advanced materials and structures under mechanical loads cannot avoid further physical processes, with the thermal transfer as the most important one, influencing their behaviour at various (macroscopic, microscopic, etc.) scales, beyond the classical approach of (linear) thermoelasticity. This approach, referring to the classical theory of thermoelasticity by [1], was developed in [2] for selected engineering applications; the mathematical existence theory, based on the convergence of Rothe sequences of approximate solutions, can be found in [3], whereas [4] pays attention to the stability of an computational algorithm, combining the method of discretization in time with the finite element method (FEM); the simplifications forcing the linearity of corresponding equations are discussed in [5].

Relative simplicity of linear formulations of thermoelastic problems opens the possibility of application of a lot of alternative approaches, as i) exploitation of exact solutions for special geometric configurations like [6], even for some carefully selected nonlinear problems by [7], ii) certain coupled differential-algebraic technique by [8], iii) the element-free Galerkin method (replacing FEM), following [9], iv) another variational approach based on decomposition of internal energy by [10], or v) the element differential method in the sense of [11]; for the review of five generalizations of the standard theory of thermoelasticity see [12].

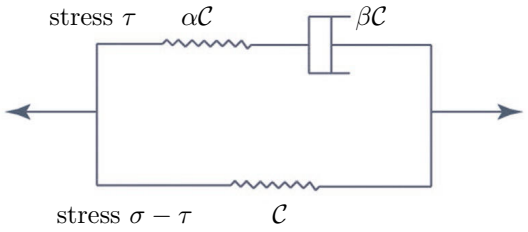

Figure 1. Scheme of the viscoelastic SLS model.

However, numerous engineering materials, namely the advanced ones, cannot be handled using the reversible purely elastic strain-stress relations; the first step taking some energy dissipation into account is the implementation of the parallel Kelvin-Voigt, or the serial Maxwell viscoelatic schemes by [13], motivated by the similar well-known schemes in the theory of electric circuits. Such generalized schemes are allowed to work with a finite number of elastic and viscous terms, discussed in [14] in terms of the generalized Kelvin chains. As a representative simple example, the so-called standard linear solid (SLS) model can be considered, see Figure 1: as the preliminary information, $\mathcal{C}$ here refers to the stiffness fourth order tensor, α, β to some additional material factors, scalar for simplicity; both stresses σ and τ are second order tensors.

Some recent approaches try to combine elastic, viscous and plastic behaviour of materials, open to the incorporation of smeared damage, cf. [15]. Unfortunately, to derive a robust, effective and inexpensive

computational algorithm covering a large range of materials with different properties, as the formation and development of ductile, brittle or quasi-brittle fracture, seems to be impossible, unlike the promising results for special problems like [16] and [17]. Namely for cement-based composites including concrete with potential reinforcement of various types, a computational model taking quite different material behaviour under tension and compression is needed. Some irreversible plastic behaviour can be added to an elastic strain-stress relation is such a case using the Rankine-Hill plastic surface by [18], generalized by [19]. Nevertheless, the so-called Mazars damage model in still most popular in engineering practice, namely the upgrade [20] of its original version [21]; some its drawbacks can be overcome by certain energy regularization, following [22]. This seems to be also reflected by the mathematical considerations of [23] where the nonlocal approach, whose basic idea comes from [24], is needed for the existence and convergence proofs, even in the quasi-static and isothermal case.

This short conference paper will demonstrate, after this first introductory section, namely the mathematical and computational aspects from this rather wide research area. The second section *Mathematical preliminaries*, containing necessary references to results and notations from functional analysis. The third section *Physical background* relies on conservation principles of classical thermomechanics, supplied with appropriate constitutive relations. In particular, *A model problem* in the fourth section pays special attention to the viscoelastic SLS model, introduced by Figure 1, and sketches the related mathematical existence and convergence analysis, based on the method of discretization in time. However, such an approach refers still to the convergence in function spaces of infinite dimensions, thus additional considerations for practical computational algorithms are needed, presented in the fifth section *Computational issues*. Available paths from the analysis of a model quasi-static linear problem to its fully dynamic modification and to some its nonlinear generalizations, required in numerous engineering computations, are shown in the sixth section *Some generalizations*; the number of still unclosed problems increases with the complexity of both physical and mathematical formulations significantly. The brief seventh section *Conclusions* summarizes the presented results with the accent to some challenges for future research.

2. Mathematical preliminaries

From the mathematical point of view, the announced analysis leads to special initial and boundary value problems for systems of partial differential equations (PDEs) of evolution. We shall consider their weak (integral) form everywhere. This needs to introduce some notations and results from the linear functional analysis, open to later generalization.

We shall work with the standard notation of (abstract) Lebesgue, Sobolev and Bochner-Sobolev function spaces, compatible with [25] and [26]. Let Ω be a domain in the 3-dimensional Euclidean space $\mathbb{R}^3$, occupied by a deformable body in its reference initial state, whose Lipchitzian boundary $\partial\Omega$ consists of two disjoint parts: Θ for the implementation of Dirichlet boundary conditions (of non-zero Hausdorff measure on $\partial\Omega$, to avoid insufficient supports) and Γ for the implementation of Neumann ones. Moreover a time interval $\mathcal{I} = [0, \mathfrak{T}]$ with a finite length $\mathfrak{T}$ is considered. The Cartesian coordinates $x = (x_1, x_2, x_3)$ will be considered in $\mathbb{R}^3$, together with the time coordinate $t \in \mathcal{I}$.

The following brief notation of function spaces will be useful:

$$\mathfrak{H} = L^2(\Omega)\,, \quad \mathcal{H} = \mathfrak{H}^3\,, \quad \mathcal{E} = L^2(\Omega)^{3\times3}_{\rm sym}\,,$$

$$\mathfrak{G} = L^2(\Gamma)\,, \quad \mathcal{G} = \mathfrak{G}^3\,, \quad \mathcal{Z} = L^\infty(\Omega)\,,$$

$$\mathcal{M} = L^\infty(\Omega)^{(3\times3)\times(3\times3)}_{\rm sym}\,, \quad \mathcal{W} = W^{1,2}(\Omega)\,,$$

$$\mathfrak{V} = \{\vartheta \in \mathcal{W}:\ \vartheta = 0 \text{ on } \Theta\}\,,$$

$$\mathcal{V} = \{v \in \mathcal{W}^3:\ v = o \text{ on } \Theta\}$$

where $o = (0,0,0)$ in $\mathbb{R}^3$, together with the brief notation of scalar products in special Hilbert spaces: $\langle .,. \rangle$ in $\mathfrak{H}$, $(.,.)$ in $\mathcal{H}$, $((.,.))$ in $\mathcal{E}$, $\langle .,. \rangle$ in $\mathcal{G}$. In addition to this notation, for any fixed $t \in \mathcal{I}$, $[v(.,t)]$ must be understood as an integral over $\tilde{t}$ from 0 to t of its argument $v(.,\tilde{t})$. Moreover the standard Hamilton (gradient) operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ will be utilized, together with the small strain operator $\varepsilon(v)$ for an appropriate function $v = (v_1, v_2, v_3)$ (virtual displacement) from $\mathcal{V}$, defined as

$$\varepsilon_{ij}(v) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$$

for any $i, j \in \{1, 2, 3\}$.

We shall see that for certain triple of unknown variables (u, τ, ϑ) mapping $\mathcal{I}$ to appropriate function spaces defined on Ω, satisfying the integral equations based on the conservation of (linear) momentum and energy, following [27], where $u = (u_1, u_2, u_3)$ means the displacement of particular points of Ω, related to the initial state $t = 0$, τ the stress part in the SLS model induced by Figure 1 and ϑ the temperature related to the initial state again, the system of 3 PDEs of evolution can be approximated by a finite number m of triples of elliptic equations, working e. g. with simple (abstract) functions $(\bar{u}^m, \bar{\tau}^m, \bar{\vartheta}^m)$ and linear Lagrange splines $(u^m, \tau^m, \vartheta^m)$. Such equations, thanks to their linearity (for a simplified model problem only, not for its later generalizations) can be then handled using the approach of [28], thanks to the knowledge of: i) the Sobolev (compact) embedding theorem, ii) the trace theorem, iii) the Gelfand triple $\mathfrak{V} \Subset \mathfrak{H} \cong \mathfrak{H}^* \Subset \mathfrak{V}^*$, with * indicating dual spaces, $\Subset$ used for compact

embeddings, $\cong$ for isomorphy between dual Hilbert spaces, iv) the Friedrichs and Korn inequalities, v) the Lax-Milgram theorem. The convergence of their solutions for $m \to \infty$, following [29], to a solution of our model problem then needs: vi) the a priori bounds for the Rothe sequences, such as those of (abstract) simple functions and linear Lagrange splines, based on the discrete Gronwall lemma, vii) the Eberlein-Shmul'yan theorem and viii) some further properties of strongly and weakly convergent sequences in related function spaces, to identify the limits of sequences generated by triples like $(\bar{u}^m, \bar{\tau}^m, \bar{\vartheta}^m)$ and $(u^m, \tau^m, \vartheta^m)$ in an appropriate sense with (u, τ, ϑ). After the loss of linearity non-trivial generalizations of i), ..., vii) are needed: namely v), coming from the Riesz representation theorem for linear functionals, in not applicable; fortunately this can be replaced by another classical existence results by [30]. This enables us to handle a wider class of (at least) quasilinear systems of PDEs, following [31] and [25], in the sixth section.

3. Physical background

In general, coupling thermal transfer with elastic, viscous, plastic, etc. deformation, open to possible incorporation of macro- and microscopic damage, relies on the principles of classical thermodynamics, supplied by some constitutive relations, containing material parameters identified by appropriate experiments. In this conference paper, due to its limited extent. we shall work only with the viscoelastic SLS model corresponding to Figure 1 and with the quasi-static time development of (u, τ, ϑ) of a deformable body, satisfying the homogeneous Cauchy initial conditions on Ω at $t = 0$

$$(u(.,0), \tau(.,0), \vartheta(.,0)) = (o, O, 0)\,,$$

O being the zero matrix in $\mathbb{R}^{3\times3}$. All material characteristics will be taken as time-independent. The weak (integral) Galerkin formulation will be preferred everywhere; its potential conversion to the variational Ritz formulation (if possible, namely for linear problems), or to the classical (differential) formulation (at least in the distributive sense) must be left to the curious reader.

The principles of conservation of (linear) momentum and energy can be then formulated as

$$((\varepsilon(v), \sigma)) = (v, f) + \langle v, g\rangle + ((\varepsilon(v), \mathcal{B}\vartheta)) \qquad (1)$$

for any virtual displacement v and

$$\langle\theta, \kappa\vartheta\rangle + [(\nabla\theta, q)] + ((\mathcal{B}\theta, \varepsilon(u))) = [\langle\theta, p\rangle] + [\langle\theta, r\rangle] \quad (2)$$

for any virtual temperature θ.

In (1) f refers to the prescribed mechanical volume loads on Ω, g to the to the prescribed mechanical surface loads on Γ and $\mathcal{B}$ involves the thermal expansion characteristics; σ, expected in $L^2(\mathcal{I}, \mathcal{E})$, remains to be specified from SLS model. To avoid technical difficulties in mathematical considerations, it is reasonable to assume $f \in L^2(\mathcal{I}, \mathcal{H})$, $g \in W^{1,2}(\mathcal{I}, \mathcal{G})$, $\mathcal{B} \in L^\infty(\Omega)^{3\times3}$ and $v \in \mathcal{V}$.

In (2) p refers to the prescribed thermal volume loads on Ω, r to the prescribed surface loads on Γ, κ denotes the thermal capacity (related to the mass unit). To avoid technical difficulties in mathematical considerations, it is reasonable to assume $p \in L^2(\mathcal{I}, \mathfrak{H})$, $r \in W^{1,2}(\mathcal{I}, \mathfrak{G})$, $\kappa \in \mathcal{Z}$ and $\theta \in \mathfrak{V}$. The thermal flux $q = (q_1, q_2, q_3)$, expected in $L^2(\mathcal{I}, \mathcal{H})$, can be evaluated as

$$q = \Lambda\nabla\vartheta\,, \qquad (3)$$

with Λ compound from the thermal conductivity components with values in $\mathbb{R}^{3\times3}$; this is well-known as the Fourier law. Its appropriate choice is $\Lambda \in L^\infty(\Omega)^{3\times3}$, moreover such a matrix should be positive definite everywhere on Ω.

To complete the system of PDEs, a relation between σ and $\varepsilon(u)$ has to be supplied. Using the approach of [23], exploiting an auxiliary variable $\tau \in L^2(\mathcal{I}, \mathcal{E})$, one can come to the additive decomposition

$$\sigma = \tau + \mathcal{C}\varepsilon(u) \qquad (4)$$

where

$$(\gamma, \mathcal{C}^{-1}\tau/\alpha) + [(\gamma, \mathcal{C}^{-1}\tau/\beta)] = (\gamma, \varepsilon(u)) \qquad (5)$$

for an arbitrary $\gamma \in \mathcal{E}$; for several alternative (both integral and differential) approaches see [14].

As evident from Figure 1, $\mathcal{C}$ (in its lower branch), needed in (4), is related to the Hooke law, well-known from the linear theory of elasticity: it is reasonable to assume $\mathcal{C} \in \mathcal{M}$ (thus $\mathcal{C}$ is allowed to contain 21 independent components). Moreover

$$\sum_{i,j,k,l=1}^{3} \mathcal{C}_{ijkl} a_{ij} a_{kl} \geq c \sum_{i,j=1}^{3} a_{ij}^2$$

should be valid for each $a \in \mathbb{R}^{3\times3}$ and for a positive constant c (almost) everywhere on Ω; this is sufficient for the construction of $\mathcal{C}^{-1}$. Let us remind that, in particluar, for isotropic materials $\mathcal{C}$ can be created using two independent Lamé factors completely (or the Young modulus and the Poisson ratio, as an alternative). Both characteristics $\alpha, \beta \in \mathcal{Z}$ will be considered as positive on Ω, to avoid the degeneration of SLS namely to the classical Hooke, Kelvin-Voigt or Maxwell models.

4. A model problem

The formulation of a model problem, represented by the system of 3 PDEs with an unknown triple of variables (u, τ, ϑ), can be now constructed i) from (1) with σ taken from (4), ii) from (2) with q taken from (3) and iii) using (5) (without changes). The results are

$$((\varepsilon(v), \mathcal{C}\varepsilon(u))) + ((\varepsilon(v), \tau)) \qquad (6)$$

$$= (v, f) + \langle v, g\rangle + ((\varepsilon(v), \mathcal{B}\vartheta))$$

for i) and

$$\langle\!(\theta, \kappa\vartheta)\!\rangle + [(\nabla\theta, \Lambda\nabla\vartheta)] + (\!(\mathcal{B}\theta, \varepsilon(u))\!) \tag{7}$$

$$= [\langle\!(\theta, p)\!\rangle] + [\langle\!(\theta, r\rangle\!\rangle]$$

for ii).

Consequently we can seek for $u \in L^2(\mathcal{I}, \mathcal{V})$, $\tau \in L^2(\mathcal{I}, \mathcal{E})$ and $\vartheta \in L^2(\mathcal{I}, \mathfrak{V})$ satisfying the system (6), (5), (7). The simplest version of the method of discretization in time, motivated by the classical Euler backward scheme, relies on the decomposition of $\mathcal{I}$ to m its equidistant subintervals $\mathcal{I}_s^m = \{t \in \mathcal{I} : (s-1)h < t \leq sh\}$, using the notation $h = \mathfrak{T}/m$: for any fixed $s \in \{1, \ldots, m\}$ we can seek, instead of (u, τ, ϑ) by (6), (5), (7), step by step, for $u_s^m \in \mathcal{V}$, $\tau_s^m \in \mathcal{E}$ and $\vartheta_s^m \in \mathfrak{V}$ satisfying

$$(\!(\varepsilon(v), \mathcal{C}\varepsilon(u_s^m))\!) + (\!(\varepsilon(v), \tau_s^m)\!) \tag{8}$$

$$= (v, f_s^m) + \langle v, g_s^m\rangle + (\!(\varepsilon(v), \mathcal{B}\vartheta_s^m)\!)\,,$$

$$(\gamma, \mathcal{C}^{-1}\tau_s^m/\alpha) \tag{9}$$

$$+h(\gamma, \mathcal{C}^{-1}\tau_1^m/\beta) + \ldots + h(\gamma, \mathcal{C}^{-1}\tau_1^m/\beta)$$

$$= (\gamma, \varepsilon(u_s^m))\,,$$

$$\langle\!(\theta, \kappa\vartheta_s^m\rangle\!\rangle \tag{10}$$

$$+h(\nabla\theta, \Lambda\nabla\vartheta_1^m) + \ldots + h(\nabla\theta, \Lambda\nabla\vartheta_s^m)$$

$$+(\!(\mathcal{B}\theta, \varepsilon(u_s^m))\!)$$

$$+h\langle\!(\theta, p_1^m)\!\rangle + \ldots + h\langle\!(\theta, p_s^m)\!\rangle + h\langle\!(\theta, r_1^m\rangle\!\rangle + \ldots + h\langle\!(\theta, r_s^m\rangle\!\rangle.$$

The difference forms of (9) and (10) (perhaps more transparent, taking all homogeneous initial conditions into account), i. e.

$$(\gamma, \mathcal{C}^{-1}(\tau_s^m - \tau_{s-1}^m)/\alpha) + h(\gamma, \mathcal{C}^{-1}\tau_s^m/\beta) \tag{11}$$

$$= (\gamma, \varepsilon(u_s^m - u_{s-1}^m))\,,$$

$$\langle\!(\theta, \kappa(\vartheta_s^m - \vartheta_{s-1}^m))\!\rangle + h(\nabla\theta, \Lambda\nabla\vartheta_s^m) \tag{12}$$

$$+(\!(\mathcal{B}\theta, \varepsilon(u_s^m - u_{s-1}^m))\!) = h\langle\!(\theta, p_s^m)\!\rangle + h\langle\!(\theta, r_s^m\rangle\!\rangle,$$

may be useful in numerous considerations, too.

Now the the proof items i), ... viii), sketched in the last paragraph of the second section, could be elaborated in details. The a priori bounds for $(u_s^m, \tau_s^m, \vartheta_s^m)$, independent of m and $s \in \{1, \ldots, m\}$, required by vi), can be obtained e. g. using certain special choices like $(v, \gamma, \theta) = (u_s^m, \tau_s^m, \vartheta_s^m)$ in (8), (11), (12). However, for practical computations it is necessary to come from $\mathcal{V}$, $\mathcal{E}$ and $\mathfrak{V}$ to some finite-dimensional spaces.

5. Computational issues

Let $\mathcal{V}^\delta$, $\mathcal{E}^\delta$ and $\mathfrak{V}^\delta$ be some finite-dimensional approximations of $\mathcal{V}$, $\mathcal{E}$ and $\mathfrak{V}$, opening the possibility of limit passage $\delta \to 0$, δ being an appropriate length characteristic for the decomposition of Ω and $\partial\Omega$, usual in the FEM techniques. In particular (but not necessarily), $\mathcal{V}^\delta$, $\mathcal{E}^\delta$ and $\mathfrak{V}^\delta$ are allowed to be linear subspaces of $\mathcal{V}$. Thus $\mathcal{V}^\delta$, $\mathcal{E}^\delta$ and $\mathfrak{V}^\delta$ have certain finite dimensions n, here considered as the same for simplicity; their bases are compound from some functions (in FEM with small supports on Ω typically) φ_I^u, φ_I^τ and φ_I^ϑ where $I \in \{1, \ldots, n\}$. Then we can replace $u_s^m(x)$ (3 components), $\tau_s^m(x)$ (6 independent components) and $\vartheta_s^m(x)$ (1 component) in (8), (11), (12) (independent of t formally) by their approximations expressed as certain sums of $u_{sI}^m\varphi_I^u(x)$ $\tau_{sI}^m\varphi_I^\tau(x)$ and $\vartheta_{sI}^m\varphi_I^\vartheta(x)$ over $I \in \{1, \ldots, n\}$. Inserting, step-by-step, $v(x) = \varphi_J^u(x)$, $\gamma(x) = \varphi_J^\tau(x)$ and $\vartheta(x) = \varphi_J^\vartheta(x)$ for $J \in \{1, \ldots, n\}$, we come to

$$\sum_{I=1}^n (\!(\varepsilon(\varphi_J^u), \mathcal{C}\varepsilon(\varphi_I^u))\!)u_{sI}^m + \sum_{I=1}^n (\!(\varepsilon(\varphi_J^u), \varphi_I^\tau)\!)\tau_{sI}^m \tag{13}$$

$$= (\varphi_J^u, f_s^m) + \langle\varphi_J^u, g_s^m\rangle + \sum_{I=1}^n (\!(\varepsilon(\varphi_J^u), \mathcal{B}\varphi_I^\vartheta)\!)\vartheta_{sI}^m\,,$$

$$\sum_{I=1}^n (\varphi_J^\tau, \mathcal{C}^{-1}\varphi_I^\tau/\alpha)(\tau_{sI}^m - \tau_{s-1\,I}^m) \tag{14}$$

$$+h\sum_{I=1}^n (\varphi_J^\tau, \mathcal{C}^{-1}\varphi_I^\tau/\beta)\tau_{sI}^m$$

$$= \sum_{I=1}^n (\varphi_J^\tau, \varepsilon(\varphi_I^u)))(u_{sI}^m - u_{s-1\,I}^m)\,,$$

$$\sum_{I=1}^n \langle\!(\varphi_J^\vartheta, \kappa\varphi_I^\vartheta)\!\rangle(\vartheta_{sI}^m - \vartheta_{s-1\,I}^m) \tag{15}$$

$$+h\sum_{I=1}^n (\nabla\varphi_J^\vartheta, \Lambda\nabla\varphi_I^\vartheta)\vartheta_{sI}^m$$

$$+\sum_{I=1}^n (\!(\mathcal{B}\varphi_J^\vartheta, \varepsilon(\varphi_I^u))\!)(u_{sI}^m - u_{s-1\,I}^m)$$

$$= h\langle\!(\varphi_J^\vartheta, p_s^m)\!\rangle + h\langle\!(\varphi_J^\vartheta, r_s^m\rangle\!\rangle,$$

Finally (8), (11), (12), for any fixed $s \in \{1, \ldots, m\}$, form a regular system of linear algebraic equations for unknown triples $(u_{sI}^m, \tau_{sI}^m, \vartheta_{sI}^m)$ with $I \in \{1, \ldots, n\}$. The recommended choice of $(\varphi_I^u, \varphi_I^\tau, \varphi_I^\vartheta)$, as well as the strategy of numerical integration, connected with the design of optimal algorithms, open to effective parallel, distributed, etc. computations, is a separate research area, exceeding the considerations of this paper.

6. Some generalizations

Several assumptions accepted in this paper were accepted to obtain sufficiently simple results and to avoid technical difficulties in the application of results of standard linear functional analysis; some examples are: i) The homogeneous Cauchy initial conditions can be replaced by more general ones; thus e. g. ϑ can be introduced as the Kelvin (absolute) temperature. ii) Material characteristics like α and β can be replaced by non-scalar (matrix, tensor, etc.) ones. However, this increases the demand on reliable identification of more material parameters in engineering problems. iii) Implementation of more types of Rothe sequences can contribute to the better knowledge of properties of solution of a slightly modified model problem, namely to those of time derivatives of u and ϑ. iv) The quasi-static formulation can be replaced by the full dynamic one, required for fast physical processes. This needs more revisions in the previous sections than i), ii), iii) because the resulting system of PDEs evolution is then piecewise parabolic and piecewise hyperbolic, but this can be still analysed without the loss of problem linearity.

Unfortunately, the design of real advanced engineering materials and structures can be rarely finished with the results from the linear theory, presented in the previous sections, regardless of some ad hoc arguments substituting proper physical, mathematical and computational analyses. Figure 2, for example, seemingly a slight modification of Figure 1, tries to incorporate some irreversible damage into the SLS model: $\mathcal{C}$ in its lower branch is replaced by $(1-\mathcal{D})\,\mathcal{C}$ where an irreversible positive smeared damage factor $\mathcal{D}$ with values between 0 and 1 (zero-valued for no damage, at $t=0$ usually), continuously increasing with the values of some strain and/or stress invariants, distinguishing between tension and compression, is implemented. The loss of linearity requires still other arguments for the existence and convergence analysis than those sketched in the second section; namely such a proper five-step algorithm for quasi-brittle composites (including concrete and similar materials), using some regularization results from [26], is presented in [32].

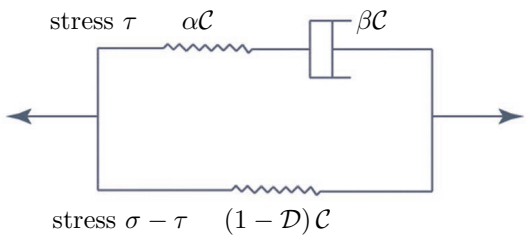

Figure 2. Scheme of the visoelastic SLS model with smeared damage.

Such an approach can be applied to the estimate of stiffness reduction due to the formation of fractured zones, but not to the initiation and development of macroscopic cracks. Such cracks can be handled as internal cohesive interfaces, with special material specific traction separation laws by [33] and [34]: the second right-hand-side additive term in (6), related to Γ, must be extended by a similar one, related to internal interfaces, where the analogy of g depends on certain jumps in normal and tangential components of u (in the sense of traces). The existence and convergence theory can rely on available results on the properties of quasilinear differential operators. Practical computations need adaptive enrichment of basis functions on and near such interfaces, with the special attention paid to the crack tip; this is, instead of the standard FEM, performed by the so-called extended FEM (XFEM), with several available upgrades and modifications.

Several further approaches try to remove ad hoc physical and/or geometrical simplifications and to replace them by a better justified considerations. Namely the thermodynamic formulation of [35] works with the specific Gibbs energy and with the complementary form of the dissipation potential where the material damaging is described by a certain separate damage tensor from $\mathcal{E}$. The specific Gibbs energy can be expressed as a function of 4 state variables: temperature, stress, damage and a set of internal parameters, related to 4 dissipative variables, both included in the formulation of a dissipative potential. Nevertheless, sufficient smoothness of a dissipative potential cannot be guaranteed to admit its partial differentiation in a reasonable sense; thus, the theory of subdifferentials must be exploited, which limits the implementation of this approach in computational algorithms, even under the small strain assumption.

To overcome selected drawbacks of small strain computations, some special FEM (or XFEM) remeshing techniques can be used, as the updated Lagrangian method by [36]. For the proper implementation of finite strain models the multiplicative decompositions of strain tensors via the theory of structured deformation, inspired by [37], are required; for its recent progress and for the attempts to collaborate with advanced physical formulations cf. [38] and [39].

7. Conclusions

The aim of this paper was to demonstrate, using a simplified model problem, the possibility of coupling thermal transfer with viscoelastic deformation using generalized Kelvin chains, namely from the mathematical and computational points of view, in the sufficiently transparent form, relying on the standard arguments from linear functional analysis. Processes and phenomena beyond this scope are referenced only in the preceding section, without more details. Nevertheless, for a rather large class of engineering problems this can be seen as a basis for the useful conference discussion on research priorities and collaboration for the several next years.

A research challenge is the formal mathematical verification of selected (in all practical cases simplified, but frequently nonlinear) engineering models, as a counterpart to their practical validation. As an

significant example, the energy dissipation (from mechanical to thermal one) during the contact / impact of several deformable bodies can be mentioned; for its preliminary analysis see [40], [41] and [42].

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