

Analysis of Dynamic Circuits in MATLAB

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Abstract — The paper deals with the proposal for the analysis of dynamic linear circuits in the MATLAB environment. A very powerful tool for the analysis of the circuits is to transform the circuits directly into the complex frequency domain using the Laplace transform and then apply the circuit analysis techniques to solve them. Applying above-mentioned methods together with the symbolic computation of MATLAB, there is no difficulty in finding the complete response of second and higher-order dynamic circuits.

Keywords — Dynamic circuit, circuit analysis, complete response, Laplace transform, symbolic computation, MATLAB.

I. INTRODUCTION

The circuit analysis is very important part of electrical engineering education at the university level. Traditional didactic methods and highly specialized software package such as PSPICE that offers flexible approach to circuit analysis and design are nowadays usually used for teaching of the circuit analysis. PSPICE is very helpful tool for the numerical circuit analysis, but it hides to students a considerable fact that a symbolic model of the circuit is behind the numerical results.

Better way to teach students the circuit analysis is to use software product that provides symbolic manipulation and can be used for symbolic circuit analysis, e.g. MATLAB, Maple, Mathematica, etc. Especially MATLAB is widespread software that is a very appropriate for lots of different university courses. The MATLAB environment is very user-friendly with powerful built-in routines that enable to solve a very wide variety of computations. Its great advantage is the fact that the user has not to spend time in learning the software but he can spend time in learning the principles of a given problem.

The analysis of dynamic circuits implies the process of finding the complete response of dynamic circuits that involves solving of first, second and higher-order circuits in the time domain or in the complex frequency domain using the Laplace transform. In the time domain, the dynamic circuits are represented by the equations in form of differential equations, and in the complex frequency domain, the dynamic circuits are represented by a set of simultaneous algebraic equations, thus the circuit analysis techniques for resistive circuits can be used. However, in the both cases the analysis of dynamic circuits is sometimes a time-consuming activity. Finding the complete response of dynamic circuits in the complex frequency domain using MATLAB greatly reduces the effort demanded to solve the set of algebraic equations and to take the inverse Laplace transform made by hand, thus there is no difficulty in solving dynamic circuits.

II. FINDING THE COMPLETE RESPONSE OF DYNAMIC CIRCUIT IN COMPLEX FREQUENCY DOMAIN

A. The Laplace Transform and the Inverse Laplace Transform

The solution of dynamic circuits in the complex frequency domain proposes two procedures. The first procedure consists of the following steps: the representation of the dynamic circuit in the time domain by differential equations and using the Laplace transform to transform the differential equations into algebraic equations. The second procedure involves the representation of the dynamic circuit in the complex frequency domain directly using the Laplace transformation and then the circuit analysis, thus in this case it will be eliminated the need to write differential equations representing the circuit.

The Laplace transform for a time-domain function $f(t)$ (it is zero for $t < 0$) of a real variable t is [1], [2]

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (1)$$

where s is a complex variable (meaning the complex frequency, $s = \sigma + j\omega$), and $F(s)$ is a complex frequency function.

For every function $f(t)$ a unique function $F(s)$ exists. This can be written as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \quad (2)$$

meaning that $f(t)$ is the inverse Laplace transform of the complex frequency function $F(s)$ and exists for $t \geq 0$.

The inverse Laplace transform is defined by the integral [1], [2]

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds \quad (3)$$

B. Complex Frequency-Domain Representation of Circuit Elements

The complex frequency-domain (s -domain) representation of circuit elements is obtained using the Laplace transform of their voltage-current relationships.

The voltage-current relationship for a linear resistor having resistance R in the s -domain [1], [2] is

$$U(s) = RI(s) \quad (4)$$

where $U(s)$ is the Laplace transform of voltage $u(t)$ across the resistor, and $I(s)$ is the Laplace transform of current through the resistor.

The voltage-current relationship for a linear inductor having inductance L in the s -domain [1], [2] is

$$U(s) = sLI(s) - Li(0) \quad (5)$$

where $U(s)$ is the Laplace transform of voltage $u(t)$ across the inductor, $I(s)$ is the Laplace transform of current through the inductor and $i(0)$ is the initial condition.

The equation (5) is used to represent the linear inductor in the s -domain as the series connection of two circuit elements that corresponds to the sum of the voltages on the right-hand side of (5).

The voltage-current relationship for a linear capacitor having capacitance C in the s -domain [1], [2] is

$$U(s) = \frac{1}{sC}I(s) + \frac{u(0)}{s} \quad (6)$$

where $U(s)$ is the Laplace transform of voltage $u(t)$ across the capacitor, $I(s)$ is the Laplace transform of current through the capacitor and $u(0)$ is the initial condition.

The equation (6) is used to represent the linear capacitor in the s -domain as the series connection of two circuit elements that corresponds to the sum of the voltages on the right-hand side of (6).

The voltage-current relationship for the coupled inductors having self-inductances L_1, L_2 and mutual inductance M in the s -domain [1], [2] is

$$\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} = \begin{bmatrix} sL_1 & \pm sM \\ \pm sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} - \begin{bmatrix} L_1 & \pm M \\ \pm M & L_2 \end{bmatrix} \begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} \quad (7)$$

where $U_1(s), U_2(s)$ is the Laplace transform of voltages $u_1(t), u_2(t)$ across the inductors L_1, L_2 respectively, $I_1(s), I_2(s)$ is the Laplace transform of currents through the inductors L_1, L_2 respectively, and $i_1(0), i_2(0)$ are the initial conditions.

Independent voltage and current sources are represented by the Laplace transform of their voltages and currents. The dependent voltage and current sources are represented by the Laplace transform of their controlled voltages and currents, and controlling voltages and currents.

The ideal operational amplifier operates in the same way in the s -domain as it does in the time domain, thus the Laplace transform of its operating conditions is applied in the s -domain.

C. Circuit Analysis Using the Laplace Transform

After replacing all the circuit elements by their s -domain representation, the s -domain representation for

the dynamic circuit with its initial conditions is obtained, and the circuit analysis techniques to solve the voltages and/or currents can be used.

Usually two systematic circuit analysis techniques are used: the mesh analysis and the node analysis. Applying one of these methods to the circuit, a set of mesh or node equations is constituted by inspection. The mesh or node equations form a square system (the number of unknown is equal to the number of equations) of simultaneous algebraic equations containing the complex frequency s , thus it can be expressed in a matrix-vector form

$$\mathbf{A}(s)\mathbf{x}(s) = \mathbf{b}(s) \quad (8)$$

where $\mathbf{A}(s)$ is the coefficient matrix, $\mathbf{x}(s)$ is a single column vector of unknowns (the mesh currents or node voltages), and $\mathbf{b}(s)$ is the single right-hand side column vector.

Solving the set of equations (8)

$$\mathbf{x}(s) = \mathbf{A}^{-1}(s)\mathbf{b}(s) \quad (9)$$

the mesh currents or node voltages, described by a ratio of polynomials in the complex variable s , are obtained.

If the mesh analysis is applied to the circuit, the s -domain element currents $\mathbf{i}(s)$ are found as linear combination of the s -domain mesh currents $\mathbf{x}(s)$

$$\mathbf{i}(s) = \mathbf{K}_c \mathbf{x}(s) \quad (10)$$

where \mathbf{K}_c is a matrix containing elements 0, +1, and -1.

The s -domain element voltages can be expressed using the Laplace transform of element voltage-current relationships.

If the node analysis is applied to the circuit, the s -domain element voltages $\mathbf{u}(s)$ are found as linear combination of the s -domain node voltages

$$\mathbf{u}(s) = \mathbf{K}_v \mathbf{x}(s) \quad (11)$$

where \mathbf{K}_v is a matrix containing elements 0, +1, and -1.

The s -domain element currents can be expressed using the Laplace transform of the element voltage-current relationship.

To get the complete response of the dynamic circuit in the time domain, it is necessary to take the inverse Laplace transform of the obtained s -domain element voltages and currents.

III. EXAMPLES OF CIRCUIT ANALYSIS USING THE LAPLACE TRANSFORM IN MATLAB

Most of the steps for finding the complete response of a dynamic circuit given above can be easily executed using the MATLAB environment.

The following text contains two representative examples of the circuit analysis using the Laplace transform in MATLAB. The analytical results of the solution together with the analysis in MATLAB and the graphical representation of the obtained results are given.

A. Analysis of Second-Order Circuit for the Overdamped, Critically Damped and Underdamped Case

The circuit shown in Fig. 1 is at steady state before the switch is changed from position 1 to position 2 at time $t = 0$.

Suppose that $u_1 = 12$ V, and $u_2 = 24$ V. Determine the element currents and voltages for $t > 0$ for three different sets of values of R_1, R_2, L and C :

- $R_1 = 10 \Omega, R_2 = 2 \Omega, L = 2$ H, $C = 1/4$ F,
- $R_1 = 12 \Omega, R_2 = 4 \Omega, L = 2$ H, $C = 1/8$ F,
- $R_1 = 2 \Omega, R_2 = 8 \Omega, L = 1$ H, $C = 1/16$ F.

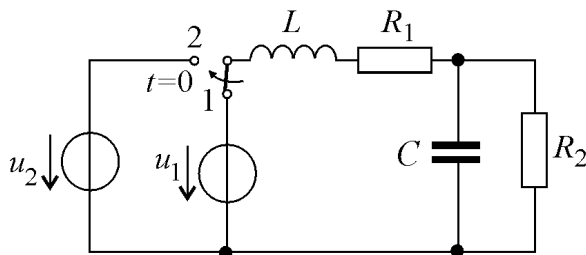


Fig. 1. Second-order circuit for the circuit analysis using the Laplace transform in MATLAB – Example A.

Since the circuit, driven by the DC voltage source, u_1 , is supposed to be at steady state before the switch is changed from position 1 to position 2, the initial conditions, the current $i(0)$ through inductor and the voltage $u(0)$ across capacitor, can be found to be:

$$i(0) = \frac{u_1}{R_1 + R_2}, \quad u(0) = \frac{R_2}{R_1 + R_2} u_1. \quad (12)$$

The s -domain representation of the circuit with the initial conditions is solved using the mesh analysis.

The mesh equations with unknowns (mesh currents) $I_1(s), I_2(s)$ can be written in a matrix-vector form:

$$\begin{bmatrix} R_1 + sL + \frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & R_2 + \frac{1}{sC} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{u_2 - u(0)}{s} + Li(0) \\ \frac{u(0)}{s} \end{bmatrix} \quad (13)$$

All the remaining computations for solving of the circuit (with given parameters) were done by running the MATLAB program (Fig. 2). The MATLAB program is a set of MATLAB built-in instructions that are saved in M-file with extension .m (a dot and suffix m).

```
% Example A
clear; clc;
syms s t
u1 = 12; u2 = 24;
for k = 1 : 3
    if k == 1
        R1 = 10; R2 = 2; L = 2; C = 1/4;
        textcase = 'over-damped case';
    elseif k == 2
        R1 = 12; R2 = 4; L = 2; C = 1/8;
        textcase = 'critically damped case';
    else
        R1 = 2; R2 = 8; L = 1; C = 1/16;
        textcase = 'under-damped case';
    end;
    iL0 = u1/(R1+R2); uC0 = iL0 * R2; % the initial conditions (Eq.12)
    A = [R1+s*L+1/(s*C) -1/(s*C); -1/(s*C) R2+1/(s*C)]; % coef. (Eq.8)
    b = [u2/s - uC0/s + L*iL0; uC0/s]; % RHS column vector (Eq.8)
    x = A \ b; % solving set of algebraic equations (Eq.9)
    is = [x(1); x(1) - x(2); x(2)]; % branch currents (Eq.10)
    us = [R1 * is(1); (s*L) * is(1) - L * iL0;... % element voltages
          (1/s/C) * is(2) + uC0/s; R2 * is(3)]; % s-domain ui relationship
    it = ilaplace(is); % inverse Laplace transform of branch currents
    ut = ilaplace(us); % inverse Laplace transform of element voltages
end;
```

Fig. 2. The MATLAB program composed for analysis of the circuit shown in Fig. 1 using the Laplace transform.

The branch currents for the critically damped case (for values of element parameters given in b) expressed in the analytical form are shown in Fig. 3.

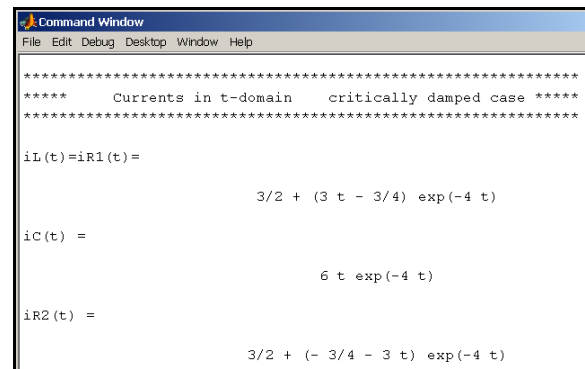


Fig. 3. The t -domain branch currents of the circuit shown in Fig. 1.

MATLAB provides many functions to represent data graphically, thus there is no problem to plot the t -domain branch currents versus time (Fig. 4).

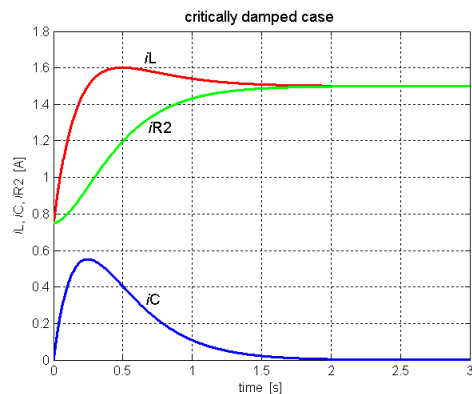


Fig. 4. The t -domain currents of the circuit shown in Fig. 1 versus time.

The graphical representation of the t -domain element voltages versus time for the critically damped case is in Fig. 5.

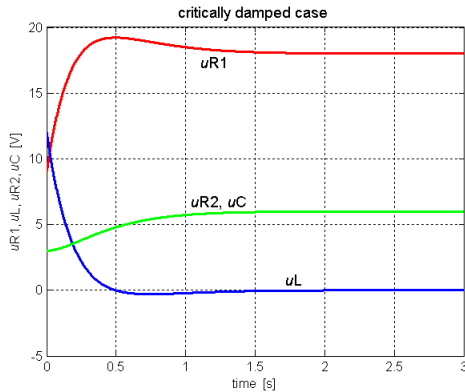


Fig. 5. The t -domain element voltages of the circuit shown in Fig. 1 versus time.

B. The Complete Response of Second-Order Circuit Containing Operational Amplifier

The circuit shown in Fig. 6 is at steady state before the input voltage $u_{in} = 1$ V at time $t = 0$ is applied. Find the output voltage $u_{out}(t)$ for $t > 0$.

Suppose that: $R_1 = 1$ k Ω , $R_2 = 200$ Ω , $C_1 = 0.125$ mF, $C_2 = 50$ μ F, and all the initial conditions are zero.

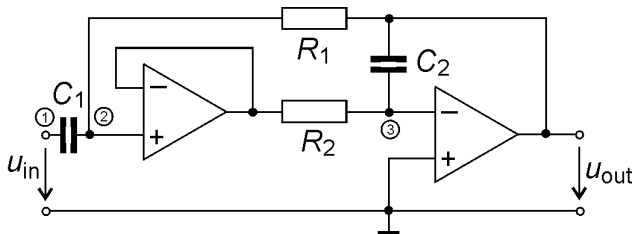


Fig. 6. Second-order circuit for the circuit analysis using the Laplace transform in MATLAB – Example B.

Since all the initial conditions are assumed zero, the complete response of the circuit is the transient response to the input unity step function applied at time $t = 0$. The s -domain representation of the circuit is solved using the node analysis.

The node equations with voltages $U_1(s), U_2(s), U_0(s)$ can be written in a matrix-vector form

$$\begin{bmatrix} 1 & 0 & 0 \\ -sC_1 & 1/R_1 + sC_1 & -1/R_1 \\ 0 & 1/R_2 & sC_2 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_0(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 0 \\ 0 \end{bmatrix} \quad (14)$$

All remaining computations for analysis of the circuit (with given parameters) were done by running the MATLAB program (Fig. 7) for the circuit analysis shown in Fig. 6 using the Laplace transform.

The time domain representation of the voltage u_{out} of the circuit for time $t > 0$ was determined in analytical form (Fig. 8).

```
% Example B
clear;clc;
syms s t
u = 1; R1 = 1e3; R2 = 200; C1 = 0.125e-3; C2 = 50e-6;
A = [1 0 0; -s*C1 1/R1+s*C1 -1/R1; 0 1/R2 s*C2]; % coef. matrix (Eq.8)
b = [u/s; 0; 0]; % RHS column vector (Eq.8)
x = A \ b; % solving the set of algebraic equations (Eq.9)
xt = ilaplace(x); % inverse laplace transform
% output: u_out output voltage in t-domain
fprintf('\n')
disp('u_out(t) = '), pretty(xt(3))
fprintf('\n')
disp('u_out(t) = '), pretty(vpa(xt(3),5))
fprintf('\n')
```

Fig. 7. The MATLAB program composed for the circuit analysis shown in Fig. 6 using the Laplace transform.

```
Command Window
File Edit Debug Desktop Window Help

u_out(t) =

- 25/7 exp(-4 t) sin(28 t)

u_out(t) =

-3.5714 exp(-4. t) sin(28. t)
```

Fig. 8. The t -domain output voltage (analytical form) of the circuit shown in Fig. 6.

In Fig. 9 the graphical representation of the voltage u_{out} of the circuit and the input voltage for time $t > 0$ is shown.

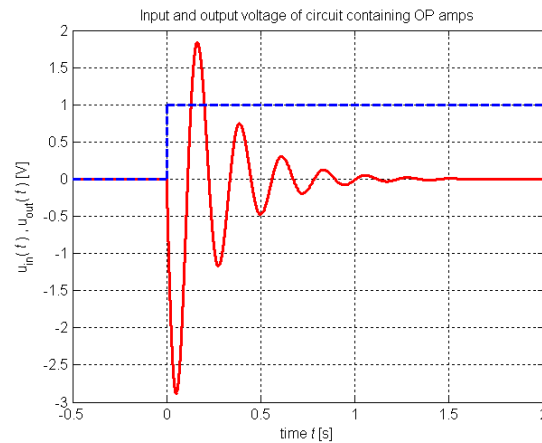


Fig. 9. The t -domain output voltage (graphical form) of the circuit shown in Fig. 6.

IV. ADVANTAGES OF PROPOSED PROCEDURE FOR ANALYSIS OF DYNAMIC CIRCUITS

Using MATLAB environment for analysis of dynamic circuits has lots of advantages but the most significant ones are the effort, time, and the analytical form of the voltages and currents in the solved circuits.

It is evident that the effort demanded to solve such type of problems is greatly reduced because it is not necessary

to solve the set of algebraic equations and to take the inverse Laplace transform.

The elapsed time that represents the processing and execution of the instructions in the composed MATLAB programs for given representative examples of the circuit analysis takes very short time. For the MATLAB program of Example A (Fig. 2) the elapsed time is 0.454 seconds, and for the MATLAB program of Example B the elapsed time is 0.359 seconds. The complete MATLAB program (all three cases with t -domain and graphical outputs) of Example A takes up 9.031 seconds only, and of Example B takes up 2.172 seconds.

The last valuable advantage of MATLAB environment is the fact that the analytical form of the voltages and currents in the t -domain are available after circuit analysis in contrast with some highly specialized software packages.

V. CONCLUSIONS

The procedure for analysis of a dynamic circuit using the Laplace transform in the MATLAB environment was proposed. It consists of the steps that must be done by hand:

- finding the initial conditions for given circuit,
- drawing the s -domain representation of the given circuit,
- applying systematic circuit analysis techniques to s -domain representation of the given circuit and writing a set of simultaneous algebraic equations containing the terms of the complex variable s ; rearranging the obtained set of equations in a matrix-vector form,

and the steps that will be a result of running the composed MATLAB program:

- assigning the values to the parameters of given circuit,
- defining the coefficient matrix and the right-hand side column vector,
- solving the set of algebraic equations,
- taking the inverse Laplace transform,
- printing the symbolic (analytic) expression of sought voltages and/or currents in a format that resembles type-set mathematics,
- plotting the graph of sought voltages and/or currents versus time.

There exists also a possibility to compose the MATLAB program that can do all the steps for the dynamic circuit analysis. In this case, the analysis techniques, which enable to formulate the circuit equations in systematic and automatic way, have to be used. Using MATLAB for solving these problems saves a lot of time and effort and everybody who needs to analyse the circuit can concentrate on gaining an insight into properties of the circuit, and making conclusions.

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