Superconformal Calogero Models as a Gauged Matrix Mechanics

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Abstract

We present basics of the gauged superfield approach to constructing the $\mathcal{N}$-superconformal multi-particle Calogero-type systems developed in arXiv:0812.4276, arXiv:0905.4951 and arXiv:0912.3508. This approach is illustrated by multi-particle systems possessing $\mathrm{SU}(1,1|1)$ and $D(2,1;\alpha)$ supersymmetries, as well as by the model of new $\mathcal{N}=4$ superconformal quantum mechanics.

1 Introduction

The celebrated Calogero model [1] is a prime example of an integrable and exactly solvable multi-particle system. It describes the system of n identical particles interacting through an inverse-square pair potential $\sum_{a\neq b} g/(x_a - x_b)^2$, $a,b = 1,\ldots,n$. The Calogero model and its generalizations provide deep connections of various branches of theoretical physics and have a wide range of physical and mathematical applications (for a review, see [2, 3]).

An important property of the Calogero model is $d = 1$ conformal symmetry $\mathrm{SO}(1,2)$. Being multi-particle conformal mechanics, this model, in the two-particle case, yields the standard conformal mechanics [4]. Conformal properties of the Calogero model and the supersymmetric generalizations of the latter give possibilities to apply them in black hole physics, since the near-horizon limits of extreme black hole solutions in M-theory correspond to AdS_2 geometry, having the same $\mathrm{SO}(1,2)$ isometry group. Analysis of the physical fermionic degrees of freedom in the black hole solutions of four- and five-dimensional supergravities shows that related $d = 1$ superconformal systems must possess $\mathcal{N}=4$ supersymmetry [5, 6, 7].

Superconformal Calogero models with $\mathcal{N}=2$ supersymmetry were considered in [8, 9] and with $\mathcal{N}=4$ supersymmetry in [10, 11, 12, 13, 14, 15]. Unfortunately, consistent Lagrange formulations for the n-particle Calogero model with $\mathcal{N}=4$ superconformal symmetry for any n is still lacking.

Recently, we developed a universal approach to superconformal Calogero models for an arbitrary number of interacting particles, including $\mathcal{N}=4$ models. It is based on the superfield gauging of some non-abelian isometries of $d = 1$ field theories [16].

Our gauge model involves three matrix superfields. One is a bosonic superfield in the adjoint representation of $\mathrm{U}(n)$. It carries the physical degrees of freedom of the superCalogero system. The second superfield is in the fundamental (spinor) representation of $\mathrm{U}(n)$ and is described by Chern-Simons mechanical action [17, 18]. The third matrix superfield accommodates the gauge "topological" supermultiplet [16]. $\mathcal{N}$-extended superconformal symmetry plays a very important role in our model. Elimination of the pure gauge and auxiliary fields gives rise to Calogero-like interactions for the physical fields.

The talk is based on the papers [19, 20, 21].

2 Gauged formulation of the Calogero model

The renowned Calogero system [1] can be described by the following action [18, 22]:

$$S_0 = \int dt \Big[\mathrm{Tr}\,(\nabla X \nabla X) + \frac{i}{2}(\bar{Z}\nabla Z - \nabla\bar{Z} Z) + c\,\mathrm{Tr}A \Big], \tag{2.1}$$

where

$$\nabla X = \dot{X} + i[A, X], \qquad \nabla Z = \dot{Z} + iAZ \qquad \nabla\bar{Z} = \dot{\bar{Z}} - i\bar{Z}A\,.$$

The action (2.1) is the action of $\mathrm{U}(n)$, $d = 1$ gauge theory. The hermitian $n\times n$-matrix field $X_a^b(t)$, $(\overline{X_a^b}) = X_b^a$, $a,b = 1,\ldots,n$ and the complex commuting $\mathrm{U}(n)$-spinor field $Z_a(t)$, $\bar{Z}^a = (\overline{Z_a})$ present the matter, scalar and spinor fields, respectively. The n^2 "gauge fields" $A_a^b(t)$, $(\overline{A_a^b}) = A_b^a$ are non-propagating ones in $d = 1$ gauge theory. The second term in the action (2.1) is the Wess-Zumino (WZ) term. The third term is the standard Fayet-Iliopoulos (FI) term.

The action (2.1) is invariant under the $d = 1$ conformal $\mathrm{SO}(1,2)$ transformations:

$$\delta t = \alpha, \qquad \delta X_a^b = \frac{1}{2}\,\dot{\alpha} X_a^b, \qquad \delta Z_a = 0, \qquad \delta A_a^b = -\dot{\alpha} A_a^b\,, \tag{2.2}$$

where the constrained parameter $\partial_t^3 \alpha = 0$ contains three independent infinitesimal constant parameters of $\mathrm{SO}(1,2)$.

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The action (2.1) is also invariant with respects to the local U(n) invariance

$$X \to gXg^\dagger, \quad Z \to gZ, \quad A \to gAg^\dagger + i\dot{g}g^\dagger\,, \qquad (2.3)$$

where $g(\tau) \in \mathrm{U}(n)$.

Let us demonstrate, in Hamiltonian formalism, that the gauge model (2.1) is equivalent to the standard Calogero system.

The definitions of the momenta, corresponding to the action (2.1),

$$\begin{aligned} P_{\scriptscriptstyle X} &= 2\nabla X\,, \qquad P_{\scriptscriptstyle Z} = \frac{i}{2}\,\bar{Z}\,, \\ \bar{P}_{\scriptscriptstyle Z} &= -\frac{i}{2}\,Z\,, \qquad P_{\scriptscriptstyle A} = 0 \end{aligned} \qquad (2.4)$$

imply the primary constraints

$$\begin{aligned} &\text{a)} \quad G \equiv P_{\scriptscriptstyle Z} - \frac{i}{2}\,\bar{Z} \approx 0\,, \quad \bar{G} \equiv \bar{P}_{\scriptscriptstyle Z} + \frac{i}{2}\,Z \approx 0\,; \\ &\text{b)} \quad P_{\scriptscriptstyle A} \approx 0 \end{aligned} \qquad (2.5)$$

and give us the following expression for the canonical Hamiltonian

$$H = \frac{1}{4}\,\mathrm{Tr}\,(P_{\scriptscriptstyle X} P_{\scriptscriptstyle X}) - \mathrm{Tr}\,(A\,T)\,, \qquad (2.6)$$

where matrix quantity T is defined as

$$T \equiv i[X, P_{\scriptscriptstyle X}] - Z \cdot \bar{Z} + cI_n\,. \qquad (2.7)$$

The preservation of the constraints (2.5b) in time leads to the secondary constraints

$$T \approx 0\,. \qquad (2.8)$$

The gauge fields A play the role of the Lagrange multipliers for these constraints.

Using canonical Poisson brackets $[X_a^b, P_{X}{}_c^d]_{\scriptscriptstyle P} = \delta_a^d \delta_c^b$, $[Z_a, P_{\scriptscriptstyle Z}^b]_{\scriptscriptstyle P} = \delta_a^b$, $[\bar{Z}^a, \bar{P}_{\scriptscriptstyle Z}{}_b]_{\scriptscriptstyle P} = \delta_b^a$, we obtain the Poisson brackets of the constraints (2.5a)

$$[G^a, \bar{G}_b]_{\scriptscriptstyle P} = -i\delta_b^a\,. \qquad (2.9)$$

Dirac brackets for these second class constraints (2.5a) eliminate spinor momenta $P_{\scriptscriptstyle Z}$, $\bar{P}_{\scriptscriptstyle Z}$ from the phase space. The Dirac brackets for the residual variables take the form

$$[X_a^b, P_{X}{}_c^d]_{\scriptscriptstyle D} = \delta_a^d \delta_c^b\,, \qquad [Z_a, \bar{Z}^b]_{\scriptscriptstyle D} = -i\,\delta_a^b\,. \qquad (2.10)$$

The residual constraints (2.8) $T = T^+$ form the $u(n)$ algebra with respect to the Dirac brackets

$$[T_a^b, T_c^d]_{\scriptscriptstyle D} = i(\delta_a^d T_c^b - \delta_c^b T_a^d) \qquad (2.11)$$

and generate gauge transformations (2.3). Let us fix the gauges for these transformations.

In the notations

$$\begin{aligned} &x_a \equiv X_a^a\,, \quad p_a \equiv P_{X}{}_a^a \quad \text{(no summation over } a\text{)}\,; \\ &x_a^b \equiv X_a^b\,, \quad p_a^b \equiv P_{X}{}_a^b \quad \text{for } a \neq b \end{aligned}$$

the constraints (2.7) take the form

$$\begin{aligned} T_a^b &= i(x_a - x_b)p_a^b - i(p_a - p_b)x_a^b + \\ &\quad i\sum_c (x_a^c p_c^b - p_a^c x_c^b) - Z_a \bar{Z}^b \approx 0 \qquad \text{for } a \neq b\,, \end{aligned} \qquad (2.12)$$

$$\begin{aligned} T_a^a &= i\sum_c (x_a^c p_c^a - p_a^c x_c^a) - Z_a \bar{Z}^a + c \approx 0 \\ &\quad \text{(no summation over } a)\,. \end{aligned} \qquad (2.13)$$

The non-diagonal constraints (2.12) generate the transformations

$$\delta x_a^b = [x_a^b, \epsilon_b^a T_b^a]_{\scriptscriptstyle D} \sim i(x_a - x_b)\epsilon_b^a\,.$$

Therefore, in case of the Calogero-like condition $x_a \neq x_b$, we can impose the gauge

$$x_a^b \approx 0\,. \qquad (2.14)$$

Then we introduce Dirac brackets for the constraints (2.12), (2.14) and eliminate x_a^b, p_a^b. In particular, the resolved expression for p_a^b is

$$p_a^b = -\frac{i}{(x_a - x_b)}\, Z_a \bar{Z}^b\,. \qquad (2.15)$$

The Dirac brackets of residual variables coincide with Poisson ones due to the resolved form of the gauge fixing condition (2.14).

After gauge-fixing (2.14), the constraints (2.13) become

$$Z_a \bar{Z}^a - c \approx 0 \qquad \text{(no summation over } a) \qquad (2.16)$$

and generate local phase transformations of Z_a. For these gauge transformations we impose the gauge

$$Z_a - \bar{Z}^a \approx 0\,. \qquad (2.17)$$

The conditions (2.16) and (2.17) eliminate Z_a and $\bar{Z}^a$ completely.

Finally, using the expressions (2.15) and the conditions (2.14), (2.16) we obtain the following expression for the Hamiltonian (2.6)

$$\begin{aligned} H_0 &= \frac{1}{4}\,\mathrm{Tr}\,(P_X P_X) = \\ &\frac{1}{4}\left(\sum_a (p_a)^2 + \sum_{a\neq b} \frac{c^2}{(x_a - x_b)^2}\right), \end{aligned} \qquad (2.18)$$

which corresponds to the standard Calogero action [1]

$$S_0 = \int \mathrm{d}t\,\Big[\sum_a \dot{x}_a \dot{x}_a - \sum_{a\neq b} \frac{c^2}{4(x_a - x_b)^2}\Big]. \qquad (2.19)$$

3 $\mathcal{N} = 2$ superconformal Calogero model

$\mathcal{N} = 2$ supersymmetric generalization of the system (2.1) is described by

- the even hermitian $(n \times n)$-matrix superfield $\mathcal{X}_a^b(t, \theta, \bar{\theta})$, $(\mathcal{X})^+ = \mathcal{X}$, $a, b = 1, \ldots, n$ [supermultiplets $(\mathbf{1}, \mathbf{2}, \mathbf{1})$];

- commuting chiral U(n)–spinor superfield $\mathcal{Z}_a(t_L,\theta)$, $\bar{\mathcal{Z}}^a(t_R,\bar\theta) = (\mathcal{Z}_a)^+$, $t_{L,R} = t \pm i\theta\bar\theta$ [supermultiplets $(\mathbf{2},\mathbf{2},\mathbf{0})$];
- commuting n^2 complex "bridge" superfields $b_a^c(t,\theta,\bar\theta)$.

The $\mathcal{N}=2$ superconformally invariant action of these superfields has the form

$$S_2 = \int dt\, d^2\theta \left[\mathrm{Tr}\left(\bar{\mathcal{D}}\mathcal{X}\,\mathcal{D}\mathcal{X}\right) + \frac{1}{2}\,\bar{\mathcal{Z}}\,e^{2V}\mathcal{Z} - c\,\mathrm{Tr}V\right]. \tag{3.1}$$

Here the covariant derivatives of the superfield $\mathscr{X}$ are

$$\mathcal{D}\mathcal{X} = D\mathcal{X} + i[\mathscr{A},\mathcal{X}]\,, \qquad \bar{\mathcal{D}}\mathcal{X} = \bar D\mathcal{X} + i[\bar{\mathscr{A}},\mathcal{X}]\,, \tag{3.2}$$

$$D = \partial_\theta + i\bar\theta\partial_t\,, \quad \bar D = -\partial_{\bar\theta} - i\theta\partial_t\,, \quad \{D,\bar D\} = -2i\partial_t\,,$$

where the potentials are constructed from the bridges as

$$\begin{aligned}\mathscr{A} &= -i\,e^{i\bar b}(De^{-i\bar b})\,,\\ \bar{\mathscr{A}} &= -i\,e^{ib}(\bar De^{-ib}) \qquad (\bar b \equiv b^+)\,.\end{aligned} \tag{3.3}$$

The gauge superfield prepotential $V_a^b(t,\theta,\bar\theta)$, $(V)^\dagger = V$, is constructed from the bridges as

$$e^{2V} = e^{-i\bar b}\,e^{ib}\,. \tag{3.4}$$

The superconformal boosts of the $\mathcal{N}=2$ superconformal group $SU(1,1|1) \simeq OSp(2|2)$ have the following realization:

$$\begin{aligned}\delta t &= -i(\eta\bar\theta + \bar\eta\theta)t\,,\\ \delta\theta &= \eta(t + i\theta\bar\theta)\,, \qquad \delta\bar\theta = \eta(t - i\theta\bar\theta)\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\delta\mathcal{X} &= -i(\eta\bar\theta + \bar\eta\theta)\,\mathcal{X}\,, \qquad \delta\mathcal{Z} = 0\,,\\ \delta b &= 0\,, \qquad \delta V = 0\,.\end{aligned} \tag{3.6}$$

Its closure with $\mathcal{N}=2$ supertranslations yields the full $\mathcal{N}=2$ superconformal invariance of the action (3.1).

The action (3.1) is invariant also with respect to the two types of the local U(n) transformations:
- τ-transformations with the hermitian $(n\times n)$-matrix parameter $\tau(t,\theta,\bar\theta) \in u(n)$, $(\tau)^+ = \tau$;
- λ–transformations with complex chiral gauge parameters $\lambda(t_L,\theta) \in u(n)$, $\bar\lambda(t_R,\theta) = (\lambda)^+$.

These U(n) transformations act on the superfields in the action (3.1) as

$$e^{ib'} = e^{i\tau}\,e^{ib}e^{-i\lambda}\,, \qquad e^{2V'} = e^{i\bar\lambda}\,e^{2V}e^{-i\lambda}\,, \tag{3.7}$$

$$\mathcal{X}' = e^{i\tau}\,\mathcal{X}\,e^{-i\tau}\,, \quad \mathcal{Z}' = e^{i\lambda}\mathcal{Z}\,, \quad \bar{\mathcal{Z}}' = \bar{\mathcal{Z}}\,e^{-i\bar\lambda}\,. \tag{3.8}$$

In terms of τ-invariant superfields V, $\mathcal{Z}$ and new hermitian $(n\times n)$-matrix superfield

$$\mathscr{X} = e^{-ib}\,\mathcal{X}\,e^{i\bar b}\,, \qquad \mathscr{X}' = e^{i\lambda}\,\mathscr{X}e^{-i\bar\lambda}\,, \tag{3.9}$$

the action (3.1) takes the form

$$S_2 = \int dt\, d^2\theta \left[\mathrm{Tr}\left(\bar{\mathscr{D}}\mathscr{X}e^{2V}\mathscr{D}\mathscr{X}e^{2V}\right) + \frac{1}{2}\,\bar{\mathcal{Z}}\,e^{2V}\mathcal{Z} - c\,\mathrm{Tr}V\right] \tag{3.10}$$

where the covariant derivatives of the superfield $\mathscr{X}$ are

$$\begin{aligned}\mathscr{D}\mathscr{X} &= D\mathscr{X} + e^{-2V}(De^{2V})\,\mathscr{X}\,,\\ \bar{\mathscr{D}}\mathscr{X} &= \bar D\mathscr{X} - \mathscr{X}e^{2V}(\bar De^{-2V})\,.\end{aligned} \tag{3.11}$$

For gauge λ-transformations we impose the WZ gauge

$$V(t,\theta,\bar\theta) = -\theta\bar\theta A(t)\,.$$

Then, the action (3.10) takes the form

$$\begin{aligned}S_2 &= S_0 + S_2^\Psi\,,\\ S_2^\Psi &= -i\,\mathrm{Tr}\int dt\,(\bar\Psi\nabla\Psi - \nabla\bar\Psi\Psi)\end{aligned} \tag{3.12}$$

where $\Psi = D\mathscr{X}|$ and

$$\nabla\Psi = \dot\Psi + i[A,\Psi]\,, \qquad \nabla\bar\Psi = \dot{\bar\Psi} + i[A,\bar\Psi]\,.$$

The bosonic core in (3.12) exactly coincides with the Calogero action (2.19).

Exactly as in the pure bosonic case, residual local U(n) invariance of the action (3.12) eliminates the nondiagonal fields X_a^b, $a{\neq}b$, and all spinor fields Z_a. Thus, the physical fields in our $\mathcal{N}=2$ supersymmetric generalization of the Calogero system are n bosons $x_a = X_a^a$ and $2n^2$ fermions Ψ_a^b. These fields present the on-shell content of n multiplets **(1,2,1)** and $n^2{-}n$ multiplets **(0,2,2)** which are obtained from n^2 multiplets **(1,2,1)** by the gauging procedure [16]. We can present it by the plot:

$$\underbrace{\mathscr{X}_a^{\;a} = (X_a^a, \Psi_a^a, C_a^a)}_{(\mathbf{1},\mathbf{2},\mathbf{1})\ multiplets} \qquad \underbrace{\mathscr{X}_a^{\;b} = (X_a^b, \Psi_a^b, C_a^b),\ {}_{a\neq b}}_{(\mathbf{1},\mathbf{2},\mathbf{1})\ multiplets}$$

$$\Downarrow \qquad \text{gauging} \qquad \Downarrow$$

$$\underbrace{\mathscr{X}_a^{\;a} = (X_a^a, \Psi_a^a, C_a^a)}_{(\mathbf{1},\mathbf{2},\mathbf{1})\ multiplets} \quad \text{interact} \quad \underbrace{\Omega_a^b = (\Psi_a^b, B_a^b, C_a^b),\ {}_{a\neq b}}_{(\mathbf{0},\mathbf{2},\mathbf{2})\ multiplets}$$

where the bosonic fields C_a^a, C_a^b and B_a^b are auxiliary components of the supermultiplets. Thus, we obtain some new $\mathcal{N}=2$ extensions of the n-particle Calogero models with n bosons and $2n^2$ fermions as compared to the standard $\mathcal{N}=2$ superCalogero with $2n$ fermions constructed by Freedman and Mende [8].

4 $\mathcal{N}=4$ superconformal Calogero model

The most natural formulation of $\mathcal{N}=4, d=1$ superfield theories is achieved in the harmonic superspace [23] parametrized by

$$\begin{aligned}&(t,\theta_i,\bar\theta^k,u_i^\pm) \sim (t,\theta^\pm,\bar\theta^\pm,u_i^\pm)\,,\\ &\theta^\pm = \theta^i u_i^\pm, \quad \bar\theta^\pm = \bar\theta^i u_i^\pm\,, \qquad i,k = 1,2.\end{aligned}$$

Commuting SU(2)-doublets $u^{\pm}_i$ are harmonic coordinates [24], subjected by the constraints $u^{+i}u^{-}_i = 1$. The $\mathcal{N} = 4$ superconformally invariant harmonic analytic subspace is parametrized by

$$(\zeta, u) = (t_A, \theta^+, \bar\theta^+, u^{\pm}_i), \qquad t_A = t - i(\theta^+\bar\theta^- + \theta^-\bar\theta^+).$$

The integration measures in these superspaces are $\mu_H = \mathrm{d}u\, \mathrm{d}t\, \mathrm{d}^4\theta$ and $\mu^{(-2)}_A = \mathrm{d}u\, \mathrm{d}\zeta^{(-2)}$.

The $\mathcal{N} = 4$ supergauge theory related to our task is described by:

- hermitian matrix superfields $\mathscr{X}(t, \theta^{\pm}, \bar\theta^{\pm}, u^{\pm}_i) = (\mathscr{X}^b_a)$ subjected to the constraints

$$\mathscr{D}^{++}\mathscr{X} = 0, \qquad \mathscr{D}^{+}\mathscr{D}^{-}\mathscr{X} = 0, \\ (\mathscr{D}^{+}\bar{\mathscr{D}}^{-} + \bar{\mathscr{D}}^{+}\mathscr{D}^{-})\mathscr{X} = 0 \tag{4.1}$$

  [multiplets **(1,4,3)**];
- analytic superfields $\mathcal{Z}^+(\zeta, u) = (\mathcal{Z}^+_a)$ subjected to the constraint

$$\mathscr{D}^{++}\mathcal{Z}^+ = 0 \tag{4.2}$$

  [multiplets **(4,4,0)**];
- the gauge matrix connection $V^{++}(\zeta, u) = (V^{++b}{}_a)$.

In (4.1) and (4.2) the covariant derivatives are defined by

$$\mathscr{D}^{++}\mathscr{X} = D^{++}\mathscr{X} + i\,[V^{++}, \mathscr{X}], \\ \mathscr{D}^{++}\mathcal{Z}^+ = D^{++}\mathcal{Z}^+ + i\,V^{++}\mathcal{Z}^+.$$

Also $\mathscr{D}^+ = D^+$, $\bar{\mathscr{D}}^+ = \bar D^+$ and the connections in $\mathscr{D}^-$, $\bar{\mathscr{D}}^-$ are expressed through derivatives of V^{++}.

The $\mathcal{N} = 4$ superconformal model is described by the action

$$S^{\alpha\neq 0}_4 = -\frac{1}{4(1+\alpha)}\int \mu_H\, \mathrm{Tr}\left(\mathscr{X}^{-1/\alpha}\right) + \\ \frac{1}{2}\int \mu^{(-2)}_A\, \mathcal{V}_0\, \widetilde{\mathcal{Z}}^+\mathcal{Z}^+ + \frac{i}{2}\,c\int \mu^{(-2)}_A\, \mathrm{Tr}\, V^{++}. \tag{4.3}$$

The tilde in $\widetilde{\mathcal{Z}}^+$ denotes 'hermitian' conjugation preserving analyticity [24, 23].

The unconstrained superfield $\mathcal{V}_0(\zeta, u)$ is a real analytic superfield, which is defined by the integral transform ($\mathscr{X}_0 \equiv \mathrm{Tr}\,(\mathscr{X})$)

$$\mathscr{X}_0(t, \theta_i, \bar\theta^i) = \\ \int \mathrm{d}u\, \mathcal{V}_0\left(t_A, \theta^+, \bar\theta^+, u^{\pm}\right)\Big|_{\theta^{\pm} = \theta^i u^{\pm}_i,\ \bar\theta^{\pm} = \bar\theta^i u^{\pm}_i}.$$

The real number $\alpha\neq 0$ in (4.3) coincides with the parameter of the $\mathcal{N} = 4$ superconformal group $D(2,1;\alpha)$ which is symmetry group of the action (4.3). Field transformations under superconformal boosts are (see the coordinate transformations in [23, 16])

$$\delta\mathscr{X} = -\Lambda_0\,\mathscr{X}, \qquad \delta\mathcal{Z}^+ = \Lambda\,\mathcal{Z}^+, \\ \delta V^{++} = 0, \tag{4.4}$$

where $\Lambda = 2i\alpha(\bar\eta^-\theta^+ - \eta^-\bar\theta^+)$, $\Lambda_0 = 2\Lambda - D^{--}D^{++}\Lambda$. It is important that just the superfield multiplier $\mathcal{V}_0$ in the action provides this invariance due to $\delta\mathcal{V}_0 = -2\Lambda\mathcal{V}_0$ (note that $\delta\mu^{(-2)}_A = 0$).

The action (4.3) is invariant under the local U(n) transformations:

$$\mathscr{X}' = e^{i\lambda}\mathscr{X}e^{-i\lambda}, \qquad \mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^+, \\ V^{++\,\prime} = e^{i\lambda}\,V^{++}\,e^{-i\lambda} - i\,e^{i\lambda}(D^{++}e^{-i\lambda}), \tag{4.5}$$

where $\lambda^b_a(\zeta, u^{\pm}) \in u(n)$ is the 'hermitian' analytic matrix parameter, $\widetilde{\lambda} = \lambda$. Using gauge freedom (4.5) we choose the WZ gauge

$$V^{++} = -2i\,\theta^+\bar\theta^+ A(t_A). \tag{4.6}$$

Considering the case $\alpha = -\dfrac{1}{2}$ (when $D(2,1;\alpha) \simeq$ OSp(4|2)) in the WZ gauge and eliminating auxiliary and gauge fields, we find that the action (4.3) has the following bosonic limit

$$S^{\alpha=-1/2}_{4,b} = \int \mathrm{d}t \left\{ \sum_a \dot x_a \dot x_a + \frac{i}{2}\sum_a (\bar Z^a_k \dot Z^k_a - \dot{\bar Z}{}^a_k Z^k_a) + \right. \\ \left. \sum_{a\neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n\,\mathrm{Tr}(\hat S\hat S)}{2(X_0)^2} \right\}, \tag{4.7}$$

where

$$(S_a)_i{}^j \equiv \bar Z^a_i Z^j_a, \quad (\hat S)_i{}^j \equiv \sum_a \left[(S_a)_i{}^j - \frac{1}{2}\delta^j_i (S_a)_k{}^k\right].$$

The fields x_a are "diagonal" fields in $X = \mathscr{X}|$. The fields Z^i define first components in $\mathcal{Z}^+$, $\mathcal{Z}^+| = Z^i u^+_i$. They are subject to the constraints

$$\bar Z^a_i Z^i_a = c \qquad \forall\, a. \tag{4.8}$$

These constraints are generated by the equations of motion with respect to the diagonal components of gauge field A.

Using Dirac brackets $[\bar Z^a_i, Z^j_b]_D = i\delta^a_b\delta^j_i$, which are generated by the kinetic WZ term for Z, we find that the quantities S_a for each a form $u(2)$ algebras

$$[(S_a)_i{}^j, (S_b)_k{}^l]_D = i\delta_{ab}\left\{\delta^l_i (S_a)_k{}^j - \delta^j_k (S_a)_i{}^l\right\}.$$

Thus modulo center-of-mass conformal potential (up to the last term in (4.7)), the bosonic limit (4.7) is none other than the integrable U(2)-spin Calogero model in the formulation of [25, 3]. Except for the case $\alpha = -\dfrac{1}{2}$, the action (4.3) yields non-trivial sigma-model type kinetic term for the field $X = \mathscr{X}|$.

For $\alpha = 0$ it is necessary to modify the transformation law of $\mathscr{X}$ in the following way [16]

$$\delta_{mod}\mathscr{X} = 2i(\theta_k\bar\eta^k + \bar\theta^k\eta_k). \tag{4.9}$$

Then the $D(2,1;\alpha=0)$ superconformal action reads

$$S_4^{\alpha=0} = -\frac{1}{4}\int \mu_H \,\mathrm{Tr}\left(e^{\mathscr{X}}\right) + \frac{1}{2}\int \mu_A^{(-2)}\, \widetilde{\mathcal{Z}}^+\mathcal{Z}^+ + \frac{i}{2}\, c\int \mu_A^{(-2)}\,\mathrm{Tr}\, V^{++}\,. \tag{4.10}$$

The $D(2,1;\alpha=0)$ superconformal invariance is not compatible with the presence of $\mathcal{V}$ in the WZ term of the action (4.10), still implying the transformation laws (4.4) for $\mathcal{Z}^+$ and for V^{++}. This situation is quite analogous to what happens in the $\mathcal{N}=2$ super Calogero model considered in Sect. 3, where the center-of-mass supermultiplet $\mathrm{Tr}(\mathscr{X})$ decouples from the WZ and gauge supermultiplets. Note that the (matrix) $\mathscr{X}$ supermultiplet interacts with the (column) $\mathcal{Z}$ supermultiplet in (3.1) and (4.10) via the gauge supermultiplet.

5 $D(2,1;\alpha)$ quantum mechanics

The $n=1$ case of the $\mathcal{N}=4$ Calogero-like model (4.3) above (the center-of-mass coordinate case) amounts to a non-trivial model of $\mathcal{N}=4$ superconformal mechanics.

Choosing the WZ gauge (4.6) and eliminating the auxiliary fields by their algebraic equations of motion, we obtain that the action takes the following on-shell form

$$S = S_b + S_f\,, \tag{5.1}$$

$$S_b = \int \mathrm{d}t\left[\dot{x}\dot{x} + \frac{i}{2}\left(\bar{z}_k\dot{z}^k - \dot{\bar{z}}_k z^k\right) - \frac{\alpha^2(\bar{z}_k z^k)^2}{4x^2} - A\left(\bar{z}_k z^k - c\right)\right], \tag{5.2}$$

$$S_f = -i\int \mathrm{d}t\left(\bar{\psi}_k\dot{\psi}^k - \dot{\bar{\psi}}_k\psi^k\right) + 2\alpha\int \mathrm{d}t\,\frac{\psi^i\bar{\psi}^k z_{(i}\bar{z}_{k)}}{x^2} + \frac{2}{3}(1+2\alpha)\int \mathrm{d}t\,\frac{\psi^i\bar{\psi}^k\psi_{(i}\bar{\psi}_{k)}}{x^2}. \tag{5.3}$$

The action (5.1) possesses $D(2,1;\alpha)$ superconformal invariance. Using the Nöther procedure, we find the $D(2,1;\alpha)$ generators. The quantum counterparts of them are

$$\mathbf{Q}^i = P\Psi^i + 2i\alpha\frac{Z^{(i}\bar{Z}^{k)}\Psi_k}{X} + i(1+2\alpha)\frac{\langle\Psi_k\Psi^k\bar{\Psi}^i\rangle}{X}\,, \tag{5.4}$$

$$\bar{\mathbf{Q}}_i = P\bar{\Psi}_i - 2i\alpha\frac{Z_{(i}\bar{Z}_{k)}\bar{\Psi}^k}{X} + i(1+2\alpha)\frac{\langle\bar{\Psi}^k\bar{\Psi}_k\Psi_i\rangle}{X}\,, \tag{5.5}$$

$$\mathbf{S}^i = -2\,X\Psi^i + t\,\mathbf{Q}^i, \quad \bar{\mathbf{S}}_i = -2\,X\bar{\Psi}_i + t\,\bar{\mathbf{Q}}_i\,. \tag{5.6}$$

$$\mathbf{H} = \frac{1}{4}P^2 + \alpha^2\frac{(\bar{Z}_k Z^k)^2 + 2\bar{Z}_k Z^k}{4X^2} - 2\alpha\frac{Z^{(i}\bar{Z}^{k)}\Psi_{(i}\bar{\Psi}_{k)}}{X^2} - (1+2\alpha)\frac{\langle\Psi_i\Psi^i\,\bar{\Psi}^k\bar{\Psi}_k\rangle}{2X^2} + \frac{(1+2\alpha)^2}{16X^2}\,, \tag{5.7}$$

$$\mathbf{K} = X^2 - t\,\frac{1}{2}\{X,P\} + t^2\,\mathbf{H}\,, \qquad \mathbf{D} = -\frac{1}{4}\{X,P\} + t\,\mathbf{H}\,, \tag{5.8}$$

$$\mathbf{J}^{ik} = i\left[Z^{(i}\bar{Z}^{k)} + 2\Psi^{(i}\bar{\Psi}^{k)}\right], \quad \mathbf{I}^{1'1'} = -i\Psi_k\Psi^k\,, \quad \mathbf{I}^{2'2'} = i\bar{\Psi}^k\bar{\Psi}_k\,, \quad \mathbf{I}^{1'2'} = -\frac{i}{2}\,[\Psi_k,\bar{\Psi}^k]\,. \tag{5.9}$$

The symbol $\langle\ldots\rangle$ denotes Weyl ordering.

It can be directly checked that the generators (5.4)–(5.9) form the $D(2,1;\alpha)$ superalgebra

$$\{\mathbf{Q}^{ai'i},\mathbf{Q}^{bk'k}\} = -2\left(\epsilon^{ik}\epsilon^{i'k'}\mathbf{T}^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}\mathbf{J}^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}\mathbf{I}^{i'k'}\right), \tag{5.10}$$

$$[\mathbf{T}^{ab},\mathbf{T}^{cd}] = -i\left(\epsilon^{ac}\mathbf{T}^{bd} + \epsilon^{bd}\mathbf{T}^{ac}\right), \tag{5.11}$$

$$[\mathbf{J}^{ij},\mathbf{J}^{kl}] = -i\left(\epsilon^{ik}\mathbf{J}^{jl} + \epsilon^{jl}\mathbf{J}^{ik}\right), \tag{5.12}$$

$$[\mathbf{I}^{i'j'},\mathbf{I}^{k'l'}] = -i\left(\epsilon^{ik}\mathbf{I}^{j'l'} + \epsilon^{j'l'}\mathbf{I}^{i'k'}\right),$$

$$[\mathbf{T}^{ab},\mathbf{Q}^{ci'i}] = i\epsilon^{c(a}\mathbf{Q}^{b)i'i}, \tag{5.13}$$

$$[\mathbf{J}^{ij},\mathbf{Q}^{ai'k}] = i\epsilon^{k(i}\mathbf{Q}^{ai'j)},$$

$$[\mathbf{J}^{i'j'},\mathbf{Q}^{ak'i}] = i\epsilon^{k'(i'}\mathbf{Q}^{aj')i}$$

due to the quantum brackets

$$[X,P] = i\,, \qquad [Z^i,\bar{Z}_j] = \delta^i_j\,, \qquad \{\Psi^i,\bar{\Psi}_j\} = -\frac{1}{2}\,\delta^i_j\,. \tag{5.14}$$

In (5.11)–(5.14) we use the notation $\mathbf{Q}^{21'i} = -\mathbf{Q}^i$, $\mathbf{Q}^{22'i} = -\bar{\mathbf{Q}}^i$, $\mathbf{Q}^{11'i} = \mathbf{S}^i$, $\mathbf{Q}^{12'i} = \bar{\mathbf{S}}^i$, $\mathbf{T}^{22} = \mathbf{H}$, $\mathbf{T}^{11} = \mathbf{K}$, $\mathbf{T}^{12} = -\mathbf{D}$.

To find the quantum spectrum, we make use of the realization

$$\bar{Z}_i = v_i^+, \qquad Z^i = \partial/\partial v_i^+ \tag{5.15}$$

for the bosonic operators where v_i^+ is a commuting complex SU(2) spinor, as well as the following realization of the odd operators

$$\Psi^i = \psi^i, \qquad \bar{\Psi}_i = -\frac{1}{2}\,\partial/\partial\psi^i\,, \tag{5.16}$$

where ψ^i are complex Grassmann variables.

The full wave function $\Phi = A_1 + \psi^i B_i + \psi^i\psi_i A_2$ is subjected to the constraints

$$\bar{Z}_i Z^i\,\Phi = v_i^+\frac{\partial}{\partial v_i^+}\,\Phi = c\,\Phi. \tag{5.17}$$

Table 1

	r_0	j	i
$A^{(c)}_{k'}(x, v^+)$	$\dfrac{\lvert\alpha\rvert(c+1)+1}{2}$	$\dfrac{c}{2}$	$\dfrac{1}{2}$
$B'^{(c)}_{k}(x, v^+)$	$\dfrac{\lvert\alpha\rvert(c+1)+1}{2} - \dfrac{1}{2}\,\mathrm{sign}(\alpha)$	$\dfrac{c}{2} - \dfrac{1}{2}$	0
$B''^{(c)}_{k}(x, v^+)$	$\dfrac{\lvert\alpha\rvert(c+1)+1}{2} + \dfrac{1}{2}\,\mathrm{sign}(\alpha)$	$\dfrac{c}{2} + \dfrac{1}{2}$	0

Requiring the wave function $\Phi(v^+)$ to be single-valued gives rise to the condition that positive constant c is integer, $c \in \mathbb{Z}$. Then (5.17) implies that the wave function $\Phi(v^+)$ is a homogeneous polynomial in v^+_i of the degree c:

$$\begin{aligned} \Phi &= A^{(c)}_1 + \psi^i B^{(c)}_i + \psi^i \psi_i A^{(c)}_2 \,, && (5.18)\\ A^{(c)}_{i'} &= A_{i', k_1 \ldots k_c} v^{+k_1} \ldots v^{+k_c} \,, && (5.19)\\ B^{(c)}_i &= B'^{(c)}_i + B''^{(c)}_i = && (5.20)\\ & \quad v^+_i B'_{k_1 \ldots k_{c-1}} v^{+k_1} \ldots v^{+k_{c-1}} + \\ & \quad B''_{(ik_1 \ldots k_c)} v^{+k_1} \ldots v^{+k_c} \,. \end{aligned}$$

On the physical states (5.17), (5.18) the Casimir operator takes the value

$$\mathbf{C}_2 = \mathbf{T}^2 + \alpha\, \mathbf{J}^2 - (1+\alpha)\, \mathbf{I}^2 + \frac{i}{4}\, \mathbf{Q}^{ai'i} \mathbf{Q}_{ai'i} = \alpha(1+\alpha)(c+1)^2/4 \,. \qquad (5.21)$$

On the same states, the Casimir operators of the bosonic subgroups SU(1,1), $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_L$,

$$\mathbf{T}^2 = r_0(r_0 - 1)\,, \qquad \mathbf{J}^2 = j(j+1)\,, \qquad \mathbf{I}^2 = i(i+1)\,,$$

take the values listed in the Table 1.

The fields B'_i and B''_i form doublets of $\mathrm{SU}(2)_R$ generated by $\mathbf{J}^{ik}$, whereas the component fields $A_{i'} = (A_1, A_2)$ form a doublet of $\mathrm{SU}(2)_L$ generated by $\mathbf{I}^{i'k'}$.

Each of $A_{i'}$, B'_i, B''_i carries a representation of the SU(1,1) group. Basis functions of these representations are eigenvectors of the generator $\mathbf{R} = \frac{1}{2}\left(a^{-1}\mathbf{K} + a\mathbf{H}\right)$, where a is a constant of the length dimension. These eigenvalues are $r = r_0 + n$, $n \in \mathbb{N}$.

6 Outlook

In [19, 20, 21], we proposed a new gauge approach to the construction of superconformal Calogero-type systems. The characteristic features of this approach are the presence of auxiliary supermultiplets with WZ type actions, the built-in superconformal invariance and the emergence of the Calogero coupling constant as a strength of the FI term of the U(1) gauge (super)field.

We see continuation of the researches presented in the solution of some problems, such as

- An analysis of possible integrability properties of new superCalogero models with finding-out a role of the contribution of the center of mass in the case of $D(2,1;\alpha)$, $\alpha \neq 0$, invariant systems.
- Construction of quantum $\mathcal{N} = 4$ superconformal Calogero systems by canonical quantization of systems (4.3) and (4.10).
- Obtaining the systems, constructed from mirror supermultiplets and possessing $D(2,1;\alpha)$ symmetry, after use gauging procedures in bi-harmonic superspace [26].
- Obtaining other superextensions of the Calogero model distinct from the A_{n-1} type (related to the root system of the SU(n) group), by applying the gauging procedure to other gauge groups.

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