

COULOMB SCATTERING IN NON-COMMUTATIVE QUANTUM MECHANICS

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ABSTRACT. Recently we formulated the Coulomb problem in a rotationally invariant NC configuration space specified by NC coordinates x_i , $i = 1, 2, 3$, satisfying commutation relations $[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k$ (λ being our NC parameter). We found that the problem is exactly solvable: first we gave an exact simple formula for the energies of the negative bound states $E_n^\lambda < 0$ (n being the principal quantum number), and later we found the full solution of the NC Coulomb problem. In this paper we present an exact calculation of the NC Coulomb scattering matrix $S_j^\lambda(E)$ in the j -th partial wave. As the calculations are exact, we can recognize remarkable non-perturbative aspects of the model: 1) energy cut-off — the scattering is restricted to the energy interval $0 < E < E_{\text{crit}} = 2/\lambda^2$; 2) the presence of two sets of poles of the S-matrix in the complex energy plane — as expected, the poles at negative energy $E_{\lambda n}^I = E_n^\lambda$ for the Coulomb *attractive* potential, and the poles at ultra-high energies $E_{\lambda n}^{II} = E_{\text{crit}} - E_n^\lambda$ for the Coulomb *repulsive* potential. The poles at ultra-high energies disappear in the commutative limit $\lambda \rightarrow 0$.

KEYWORDS: Coulomb scattering, non-commutativity, quantum mechanics.

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1. INTRODUCTION

The basic ideas of non-commutative geometry have been developed in [1] and in a form of matrix geometry in [2]. The analysis performed in [3] led to the conclusion that quantum vacuum fluctuations and Einstein gravity could create (micro)black holes which prevent localization of space-time points. Mathematically this requires non-commutative (NC) coordinates x^μ in space-time satisfying specific commutation relations, e.g. Heisenberg-Moyal commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \mu, \nu = 0, 1, 2, 3, \quad (1)$$

where $\theta^{\mu\nu}$ are given numerical constants that specify the non-commutativity of the space-time in question. Later in [4] it was shown that such field theories in NC spaces can emerge as effective low energy limits of string theories. These results supported a vivid development of non-commutative QFT. However, such models are very complicated and contain various unpleasant and unwanted features.

However, it may be interesting to reverse the approach. Not to use the NC geometry to improve the foundations of QFT, but to test the effect of non-commutativity of the space on well-defined problems in quantum mechanics (QM), such as the harmonic oscillator, the Aharonov-Bohm effect, the Coulomb problem and the planar spherical well, see e.g. [5–7].

Recently, in [9, 10] we formulated the Coulomb problem in a rotationally invariant NC configuration space \mathbf{R}_λ^3 specified by NC coordinates x_k , $k = 1, 2, 3$,

satisfying the commutation relations

$$[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k, \quad (2)$$

where λ is a parameter of the non-commutativity with the dimension of length. We found the model exactly solvable. In [9] we gave an exact simple formula for the NC negative bound state energies, and in [10] we presented the full solution of the NC Coulomb problem. In this paper we present exact formulas for the NC Coulomb S-matrix in the j -th partial wave. A similar construction of a 3D noncommutative space, as a sequence of fuzzy spheres, was proposed in [11]. However, various fuzzy spheres are related to each other differently there (not leading to the flat 3D geometry at large distances).

This paper is organized as follows. In Section 2 we provide the formulation of the Coulomb problem in NC space and we briefly describe the method of solution suggested in [9] and [10]. In Section 3 we sketch the determination of the Coulomb S-matrix in spherical coordinates within standard QM (see [12]), and then we generalize the calculations to the non-commutative context. As the calculations are exact, we shall be able to recognize remarkable non-perturbative aspects of the NC Coulomb problem:

- the cut-off for the scattering energy $E \in (0, E_{\text{crit}})$, where $E_{\text{crit}} = 2/\lambda^2$;
- two sets of S-matrix poles: poles at energies $E < 0$ for *attractive* Coulomb potential and poles at

ultra-high energies $E > E_{\text{crit}}$ for *repulsive* Coulomb potential that disappear in the commutative limit $\lambda \rightarrow 0$.

In Section 4 we provide a brief discussion and conclusions.

2. NON-COMMUTATIVE QUANTUM MECHANICS

2.1. NON-COMMUTATIVE CONFIGURATION SPACE

We realize the NC coordinates in \mathbf{R}_λ^3 , similarly as the Jordan-Wigner realization of the fuzzy sphere in [8], in terms of 2 pairs of boson annihilation and creation operators $a_\alpha, a_\alpha^\dagger, \alpha = 1, 2$, satisfying the following commutation relations:

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0. \quad (3)$$

They act in an auxiliary Fock space \mathcal{F} spanned by normalized vectors

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0\rangle. \quad (4)$$

Here $|0\rangle \equiv |0, 0\rangle$ denotes the normalized vacuum state: $a_1|0\rangle = a_2|0\rangle = 0$.

The noncommutative coordinates $x_j, j = 1, 2, 3$, in the space \mathbf{R}_λ^3 satisfying (2) are given as

$$x_j = \lambda a^+ \sigma_j a \equiv \lambda \sigma_{\alpha\beta}^j a_\alpha^\dagger a_\beta, j = 1, 2, 3, \quad (5)$$

where λ is the universal length parameter and σ_j are Pauli matrices. The operator that approximates the NC analog of the Euclidean distance from the origin is

$$r = \rho + \lambda, \rho = \lambda N, N = a_\alpha^\dagger a_\alpha. \quad (6)$$

It can easily be shown that $[x_i, r] = 0$, and $r^2 - x_j^2 = \lambda^2$. A strong argument supporting the exceptional role of r will be given later.

2.2. HILBERT SPACE \mathcal{H}_λ OF NC WAVE FUNCTIONS

Let us consider a linear space spanned by normal ordered polynomials containing the same number of creation and annihilation operators:

$$\Psi = (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} (a_1)^{n_1} (a_2)^{n_2}, m_1 + m_2 = n_1 + n_2. \quad (7)$$

\mathcal{H}_λ is our denotation for the Hilbert space of linear combinations of functions (7) closed with respect to the norm

$$\|\Psi\|^2 = 4\pi\lambda^3 \text{Tr}[(N+1)\Psi^\dagger\Psi] = 4\pi\lambda^2 \text{Tr}[r\Psi^\dagger\Psi]. \quad (8)$$

The rotationally invariant weight $w(r) = 4\pi\lambda^2 r$ is determined by the requirement that a ball in \mathbf{R}_λ^3 with radius $r = \lambda(N+1)$ should possess a standard volume in the limit $r \rightarrow \infty$. It can be shown that the chosen weight $w(r)$ guarantees that the ball in question has the volume $V_r = \frac{4\pi}{3} r^3 + o(\lambda)$.

Remark. The weighted trace $\text{Tr}[w(r)\dots]$ with $w(r) = 4\pi\lambda^2 r$ goes to the usual volume integral $\int d^3\vec{x}\dots$ at large distances. The 3D non-commutative space proposed in [11] corresponds to the choice $w(\hat{r}) = \text{const}$ and at large distances does not correspond to the flat space \mathbf{R}^3 .

2.3. ORBITAL MOMENTUM IN \mathcal{H}_λ

In \mathcal{H}_λ we define orbital momentum operators, the generators of rotations L_j , as follows:

$$L_j \Psi = \frac{1}{2} [a^+ \sigma_j a, \Psi], j = 1, 2, 3. \quad (9)$$

They are hermitian (self-adjoint) operators in \mathcal{H}_λ and obey the standard commutation relations

$$[L_i, L_j] \Psi \equiv (L_i L_j - L_j L_i) \Psi = i \varepsilon_{ijk} L_k \Psi. \quad (10)$$

The standard eigenfunctions $\Psi_{jm}, j = 0, 1, 2, \dots, m = -j, \dots, +j$, satisfying

$$L_i^2 \Psi_{jm} = j(j+1) \Psi_{jm}, L_3 \Psi_{jm} = m \Psi_{jm}, \quad (11)$$

are given by the formula

$$\Psi_{jm} = \sum_{(jm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} R_j(\varrho) \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}, \quad (12)$$

where $\varrho = \lambda a_\alpha^\dagger a_\alpha = \lambda N$. The summation goes over all nonnegative integers satisfying $m_1 + m_2 = n_1 + n_2 = j, m_1 - m_2 - n_1 + n_2 = 2m$. For any fixed $R_j(\varrho)$ equation (12) defines a representation space for a unitary irreducible representation with spin j .

2.4. THE NC ANALOG OF LAPLACE OPERATOR IN \mathcal{H}_λ

We postulate the NC analog of the usual Laplace operator in the form:

$$\begin{aligned} \Delta_\lambda \Psi &= -\frac{1}{\lambda r} [a_\alpha^\dagger, [a_\alpha, \Psi]] \\ &= \frac{1}{\lambda^2(N+1)} [a_\alpha^\dagger, [a_\alpha, \Psi]]. \end{aligned} \quad (13)$$

This choice is motivated by the following facts:

- (1.) a double commutator is an analog of a second order differential operator;
- (2.) factor r^{-1} guarantees that the operator Δ_λ is hermitian (self-adjoint) in \mathcal{H}_λ , and finally,
- (3.) factors λ^{-1} or λ^{-2} respectively, guarantee the correct physical dimension of Δ_λ and its non-trivial commutative limit.

Calculating the action of (13) on Ψ_{jm} given in (12) we can check whether the postulate (13) is a reasonable choice. First, we represent the operator $R_j(\varrho)$ in (12) as a normal ordered form of an analytic function $\mathcal{R}_j(\varrho)$:

$$\begin{aligned} R_j(\varrho) &= : \mathcal{R}_j(\varrho) : = \sum_k c_k^j \cdot \varrho^k: \\ &= \sum_k c_k^j \lambda^k \frac{N!}{(N-k)!} \end{aligned} \quad (14)$$

The last equality follows from the formula

$$:N^k:|n_1, n_2\rangle = \frac{n!}{(n-k)!}|n_1, n_2\rangle, n = n_1 + n_2 \quad (15)$$

(which can be proved by induction in k). Now we will use the following commutation relations

$$\begin{aligned} [a_\alpha^\dagger, :N^k:] &= -ka_\alpha^\dagger :N^{k-1}: \Rightarrow [a_\alpha^\dagger, :\mathcal{R}:] = -\lambda a_\alpha^\dagger :\mathcal{R}':, \\ [a_\alpha, :N^k:] &= k:N^{k-1}:a_\alpha \Rightarrow [a_\alpha, :\mathcal{R}:] = \lambda :\mathcal{R}':a_\alpha, \end{aligned} \quad (16)$$

where \mathcal{R}' denotes the derivative of \mathcal{R} : $\mathcal{R}' = \sum_{k=1}^\infty kc_k \varrho^{k-1}$. Using (16), the following formula can be derived:

$$\begin{aligned} [a_\alpha^\dagger, [a_\alpha, \Psi]] &= \sum_{(jm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} \\ \times :[-\varrho \mathcal{R}''(\varrho) - 2(j+1)\mathcal{R}'(\varrho)] &: \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}. \end{aligned} \quad (17)$$

where $\mathcal{R}''(\varrho)$ is defined as the derivative of $\mathcal{R}'(\varrho)$. In the commutative limit $\lambda \rightarrow 0$ the operator ϱ formally reduces to the usual radial r variable in \mathbf{R}^3 , and we see that Δ_λ just reduces to the standard Laplace operator in \mathbf{R}^3 .

2.5. THE POTENTIAL TERM IN \mathcal{H}_λ

The operator V corresponding to a central potential in QM is defined simply as the multiplication of the NC wave function by $V(r)$:

$$(V\Psi)(r) = V(r)\Psi = \Psi V(r). \quad (18)$$

In the commutative case the Coulomb potential $\Phi(r) = -\frac{q}{r}$ is the radial solution of the equation

$$\Delta\Phi(r) = 0 \quad (19)$$

vanishing at infinity. Due to our choice of the NC Laplace operator Δ_λ the NC analog of this equation is

$$\Delta_\lambda\Phi(r) = 0 \iff [a_\alpha^\dagger, [a_\alpha, \Phi(N)]] = 0.$$

The last equation can be rewritten as a simple recurrent relation

$$\begin{aligned} (N+2)\Phi(N+1) - (N+1)\Phi(N) \\ = (N+1)\Phi(N) - N\Phi(N-1) \end{aligned} \quad (20)$$

that can be easily solved. Its solution vanishing at infinity is given as

$$\Phi(N) = -\frac{q'}{N+1} \iff \Phi(r) = -\frac{q}{r}. \quad (21)$$

We identify $\Phi(r)$ with the NC analog of the Coulomb potential. We see that the $1/r$ dependence of the NC Coulomb potential is inevitable.

3. THE COULOMB PROBLEM IN NC QM

3.1. NC RADIAL SCHRÖDINGER EQUATION

Based on (13) and (21) we postulate the NC analog of the Schrödinger equation with the Coulomb potential in \mathbf{R}_λ^3 as

$$\begin{aligned} \frac{\hbar^2}{2m\lambda r} [a_\alpha^\dagger, [a_\alpha, \Psi]] - \frac{q}{r}\Psi = E\Psi \\ \iff \frac{1}{\lambda} [a_\alpha^\dagger, [a_\alpha, \Psi]] - 2\alpha\Psi = k^2 r\Psi, \end{aligned} \quad (22)$$

where q is a square of electric charge $q = \pm e^2$ ($q > 0$ or $q < 0$ corresponding to the Coulomb attraction or repulsion respectively), $\alpha = mq/\hbar^2$ and $k^2 = 2mE/\hbar^2$.

Putting $\Psi = \Psi_{jm}$ given in (12) into NC Schrödinger equation (22) we come to the radial Schrödinger equation for $R_j = :\mathcal{R}:$. Using (17) and the relation

$$\begin{aligned} r\Psi_{jm} = \sum_{(jm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} \\ \times :[(\varrho + \lambda j + \lambda)\mathcal{R}_j + \lambda\varrho\mathcal{R}'_j]: \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}, \end{aligned} \quad (23)$$

we obtain

$$\begin{aligned} :\varrho\mathcal{R}''_j + [k^2\lambda\varrho + 2j + 2]\mathcal{R}'_j \\ + [k^2\varrho + k^2\lambda(j+1) + 2\alpha]\mathcal{R}_j: = 0. \end{aligned} \quad (24)$$

We claim (24) to be an NC analog of the usual radial Schrödinger equation known from the standard QM. There definitely is a resemblance, as in the limit $\lambda \rightarrow 0$ the terms in (24) proportional to λ representing the NC corrections disappear. Considering the same limit we see that we also do not need to worry about the colon marks denoting the normal ordering, since for zero λ it makes no difference whatsoever whether we care for the ordering or not.

Now we can solve the NC radial Schrödinger equation in two separate steps:

- (1.) We associate the following ordinary differential equation to the mentioned operator radial Schrödinger equation (24):

$$\begin{aligned} \varrho\mathcal{R}''_j + [k^2\lambda\varrho + 2j + 2]\mathcal{R}'_j \\ + [k^2\varrho + k^2\lambda(j+1) + 2\alpha]\mathcal{R}_j = 0, \end{aligned} \quad (25)$$

with ϱ being real variable, and we will solve this one. But why do we expect this step to be of any use to us, when we actually *do* have to care about the ordering? The key information follows from (16): the derivatives of \mathcal{R} appearing in (17) are just like carbon copies of the usual derivatives.

- (2.) Now bearing this in mind, we put $R = :\mathcal{R}:$, the solution of (24), to be of the same form as \mathcal{R} , the solution of (25), except that $\varrho = \lambda N$ and the normal

powers $:\varrho^n$: have to be calculated. Fortunately there is a simple formula relating the two, namely

$$\begin{aligned} :\varrho^n: &= \lambda^n :N^n: = \lambda^n \frac{N!}{(N-n)!} \\ :\varrho^{-n}: &= \lambda^{-n} :N^n: = \lambda^{-n} \frac{N!}{(N-n)!} \end{aligned} \quad (26)$$

All we need is to rewrite $:\mathcal{R}$: using those relations. Then the comparison of QM and NCQM will be at hand.

3.2. COULOMB SCATTERING IN QM

To begin with, we briefly sum up the QM results before handling our NCQM case. The solution of the radial Schrödinger equation for a particle in the potential $V(r) = -\alpha/r$ with the angular momentum j and energy $E > 0$ regular in $r \rightarrow 0$ is given as

$$\begin{aligned} R_j^{QM} &= e^{ikr} \phi\left(j+1 - i\frac{\alpha}{k}, 2j+2, -2ikr\right), \\ k &= \sqrt{2E} > 0, \end{aligned} \quad (27)$$

in terms of the confluent hypergeometric function (see [12]):

$$\phi(a, c, z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{z^m}{m!}. \quad (28)$$

Here $(a)_m$ is the so-called Pochhammer symbol: $(a)_m = a(a+1) \cdots (a+m-1)$, $m = 0, 1, 2, \dots$, and $(a)_0 = 1$. In (27) we have refrained from writing down m/\hbar^2 explicitly. This will simplify the formulas and will not do any harm, since the full form can be restored anytime.

The solution (27) is real and for $r \rightarrow \infty$ it can be written as the sum of two complex conjugated parts corresponding to an in- and out-going spherical wave. In the following formula a real factor common for both parts is left out, having no influence on the S-matrix.

$$\begin{aligned} R_j^{QM} &\sim \frac{i^{j+1}}{\Gamma(j+1 + i\frac{\alpha}{k})} e^{ikr + i\frac{\alpha}{k} \ln(2kr)} \\ &+ \frac{i^{-j-1}}{\Gamma(j+1 - i\frac{\alpha}{k})} e^{-ikr - i\frac{\alpha}{k} \ln(2kr)}. \end{aligned} \quad (29)$$

The S-matrix for the j -th partial wave is defined as the ratio of the r -independent factors multiplying the exponentials with the kinematical factor $(-1)^{j+1}$ left out:

$$\begin{aligned} S_j^{QM}(E) &= \frac{\Gamma(j+1 - i\frac{\alpha}{k})}{\Gamma(j+1 + i\frac{\alpha}{k})}, \\ E &= \frac{1}{2} k^2 > 0. \end{aligned} \quad (30)$$

3.3. COULOMB SCATTERING IN NCQM

Now let us have a look on the Coulomb scattering in NCQM. The solution of equation (25) regular at the

origin is again given in terms of confluent hypergeometric function (see [10]):

$$\begin{aligned} \mathcal{R}_{j\pm} &= \exp[(\pm\pi(E) - \lambda E)\varrho] \\ &\times \phi\left(j+1 \pm \frac{\alpha}{\pi(E)}, 2j+2, \mp 2\pi(E)\varrho\right), \end{aligned} \quad (31)$$

where

$$\pi(E) = \sqrt{2E\left(\frac{1}{2}\lambda^2 E - 1\right)}. \quad (32)$$

We point out that both $\mathcal{R}_{j+} = \mathcal{R}_{j-}$ due to the Kummer identity valid for confluent hypergeometric function (see [13]).

Scattering solutions, containing in- and out-going spherical waves, can be obtained only for energy properly restricted to the values

$$2E\left(\frac{1}{2}\lambda^2 E - 1\right) < 0 \iff E \in (0, 2/\lambda^2). \quad (33)$$

Thus we recovered energy cut-off $E_{\text{crit}} = 2/\lambda^2$. For $E \in (0, E_{\text{crit}})$ we put

$$\pi(E) = ip, \quad p = \sqrt{2E\left(1 - \frac{1}{2}\lambda^2 E\right)} > 0, \quad (34)$$

and chose the solution (31) as $\mathcal{R}_{Ej} = \mathcal{R}_{Ej+}$; we labeled the solution by the admissible value of energy $E \in (0, E_{\text{crit}})$. The solution of the NC radial Schrödinger equation (24) is $R_{Ej} = :\mathcal{R}_{Ej}:$. Using (26) the calculation is straightforward and we find for R_{Ej} the expression

$$\begin{aligned} R_{Ej} &= \left(\frac{p + i\lambda E}{p - i\lambda E}\right)^N \\ &\times F\left(j+1 - i\frac{\alpha}{p}, -N, 2j+2; 2i\lambda p \frac{p - i\lambda E}{p + i\lambda E}\right) \end{aligned} \quad (35)$$

in terms of the usual hypergeometric function

$$F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}. \quad (36)$$

The radial dependence of R_j is present in the hermitian operator N : $r = \varrho + \lambda$, $\varrho = \lambda N$.

By analogy with (29) we will rewrite also the NC solution as a sum of two hermitian conjugated terms corresponding to the in- and out-going spherical wave. First, we express R_{Ej} as

$$R_{Ej} = (1-z)^{b/2} F(a, b, c; z). \quad (37)$$

According to Kummer identities (see [13]), $F(a, b, c; z)$ can be written as a linear combination of two other solutions of the hypergeometric equation, namely

$$(-z)^{-a} F(a, a+1-c, a+1-b; z^{-1})$$

and

$$(z)^{a-c} (1-z)^{c-a-b} F(c-a, 1-a, c+1-a-b; \frac{z-1}{z}).$$

Again, leaving out the common hermitian factor which is irrelevant regarding the S -matrix, we can write:

$$\begin{aligned}
R_{Ej} &\sim \frac{(-1)^{j+1} e^{\alpha\pi/p}}{\Gamma(j+1+i\frac{\alpha}{p})} \left(\frac{p+i\lambda E}{p-i\lambda E} \right)^{N+j+1-i\frac{\alpha}{p}} \\
&\times \frac{\Gamma(2j+2)\Gamma(N+1)}{\Gamma(N+2+j-i\alpha/p)} (2\lambda p)^{-1-j+i\frac{\alpha}{p}} \\
&\times F\left(A, B, C; -\frac{i}{2\lambda p} \left(\frac{p+i\lambda E}{p-i\lambda E} \right)\right) \\
&+ \frac{(-1)^j e^{\alpha\pi/p}}{\Gamma(j+1-i\frac{\alpha}{p})} \left(\frac{p-i\lambda E}{p+i\lambda E} \right)^{N+j+1+i\frac{\alpha}{p}} \\
&\times \frac{\Gamma(2j+2)\Gamma(N+1)}{\Gamma(N+2+j+i\alpha/p)} (2\lambda p)^{-1-j-i\frac{\alpha}{p}} \\
&\times F\left(A^*, B^*, C^*; \frac{i}{2\lambda p} \left(\frac{p-i\lambda E}{p+i\lambda E} \right)\right), \quad (38)
\end{aligned}$$

where

$$\begin{aligned}
A &= j+1-i\frac{\alpha}{p}, & B &= -j-i\frac{\alpha}{p}, \\
C &= N+2+j-i\frac{\alpha}{p}. \quad (39)
\end{aligned}$$

In the limit $r = \lambda(N+1) \rightarrow \infty$ this can be simplified as (for details see ([10])):

$$\begin{aligned}
R_{Ej} &\sim (-1)^{j+1} i^{-j-1} e^{-\alpha\pi/2p} \\
&\times \frac{\Gamma(2j+2)}{\Gamma(j+1-i\alpha/p)} \frac{e^{-i\frac{\alpha}{p} \ln(2pr)}}{(2pr)^{j+1}} \\
&\times \exp\left[-(r/\lambda + j + i\alpha/p) \ln \frac{p+i\lambda E}{p-i\lambda E}\right] \\
&\quad + (-1)^{j+1} i^{j+1} e^{-\alpha\pi/2p} \\
&\times \frac{\Gamma(2j+2)}{\Gamma(j+1+i\alpha/p)} \frac{e^{i\frac{\alpha}{p} \ln(2pr)}}{(2pr)^{j+1}} \\
&\times \exp\left[-(r/\lambda + j - i\alpha/p) \ln \frac{p-i\lambda E}{p+i\lambda E}\right]. \quad (40)
\end{aligned}$$

The S -matrix is the ratio of the r -independent factors in (40):

$$\begin{aligned}
S_j^\lambda(E) &= \frac{\Gamma(j+1-i\frac{\alpha}{p})}{\Gamma(j+1+i\frac{\alpha}{p})}, \\
E &= \frac{1}{\lambda^2} \left(1 + i\sqrt{\lambda^2 p^2 - 1} \right). \quad (41)
\end{aligned}$$

The function $E = E(p)$, given above with a positive square root in for $p \in (1/\lambda, +\infty)$, is a conformal map inverse to (34), which maps the cut p right-half-plane into the E upper-half-plane.

The physical-relevant values of the S -matrix are obtained as $S_j^\lambda(E + i\varepsilon)$ in the limit $\varepsilon \rightarrow 0_+$. The interval corresponding to the scattering $E \in (0, 2/\lambda^2)$ is mapped onto the branch cut in the p -plane as follows: The energy interval $E \in (0, 1/\lambda^2)$ maps on the upper edge of the branch cut $p \in (0, 1/\lambda)$, whereas $E \in (1/\lambda^2, 2/\lambda^2)$ maps on the upper edge of the branch cut $p \in (0, 1/\lambda)$.

3.4. COULOMB BOUND STATES AS POLES OF THE S -MATRIX

We will begin with a brief reminder of the QM case: Supposing the potential is attractive ($\alpha > 0$), the S -matrix (30) has poles in the upper complex k -plane for

$$k_n = i\frac{\alpha}{n}, \quad n = j+1, j+2, \dots \quad (42)$$

It is obvious that the bound state energy levels correspond to the poles of the S -matrix:

$$E_n = -\frac{\alpha^2}{2n^2}, \quad n = j+1, j+2, \dots \quad (43)$$

In NCQM there is an analogy, the poles of the S -matrix occur in the case of attractive potential ($\alpha > 0$) for some special values of energy below 0. However, poles can also be found in the case of repulsive potential ($\alpha < 0$) for particular values of energy above $2/\lambda^2$.

(1.) Poles of the S -matrix for attractive potential.

$$\begin{aligned}
p_n^\lambda &= i\frac{\alpha}{n} \alpha > 0 \\
\iff E_{\lambda n}^I &= \frac{1}{\lambda^2} \left(1 - \sqrt{1 + (\lambda\alpha/n)^2} \right) < 0, \\
n &= j+1, j+2, \dots \quad (44)
\end{aligned}$$

In the limit $\lambda \rightarrow 0$ this coincides with the standard self-energies (43). Let us denote

$$\kappa_n = \frac{\lambda\alpha}{n}, \quad \Omega_n^I = \frac{\kappa_n - \sqrt{1 + \kappa_n^2} + 1}{\kappa_n + \sqrt{1 + \kappa_n^2} - 1}. \quad (45)$$

Then the solution (35) is

$$R_{nj}^I = (\Omega_n^I)^N F(-n, -N, 2j+2; -2\kappa_n(\Omega_n^I)^{-1}). \quad (46)$$

It is integrable since $\Omega_n \in (0, 1)$ for positive κ_n and under given conditions the hypergeometric function is a polynomial.

(2.) Poles of the S -matrix for repulsive potential.

$$\begin{aligned}
p_n^\lambda &= i\frac{\alpha}{n} \alpha < 0 \\
\iff E_{\lambda n}^{II} &= \frac{1}{\lambda^2} \left(1 + \sqrt{1 + (\lambda\alpha/n)^2} \right) > 2/\lambda^2, \\
n &= j+1, j+2, \dots \quad (47)
\end{aligned}$$

Now (35) has the form

$$R_{nj}^{II} = (-\Omega_n^{II})^N F(-n, -N, 2j+2; 2\kappa_n(\Omega_n^{II})^{-1}), \quad (48)$$

where

$$\Omega_n^{II} = -\frac{\kappa_n + \sqrt{1 + \kappa_n^2} + 1}{\kappa_n - \sqrt{1 + \kappa_n^2} - 1}. \quad (49)$$

The definition of κ_n is the same as in (45) (note that it is negative this time). Since $\Omega_n^{II} = \Omega_n^I \in (0, 1)$ the solution (35) is integrable because the hypergeometric function terminates as in the previous case. These states disappear from the Hilbert space in the limit $\lambda \rightarrow 0$.

4. CONCLUSIONS

In this paper we have investigated the Coulomb scattering in NCQM in the framework of the model formulated in [9] and [10]. As the model is exactly solvable, we were able to find an exact formula for the NC Coulomb scattering matrix $S_j^\lambda(E)$ in j -th partial wave. We found that it turns out to have remarkable non-perturbative aspects:

- (1.) Energy cut-off — the scattering is restricted to the energy interval $0 < E < E_{\text{crit}} = 2/\lambda^2$;
- (2.) $S_j^\lambda(E)$ has *two* sets of poles in the complex energy plane: as expected, the poles at negative energy $E_{\lambda n}^{\text{I}} < 0$ for *attractive* Coulomb potential that reduce to the standard H-atom bound states energies in the commutative limit $\lambda \rightarrow 0$, and poles at ultra-high energies $E_{\lambda n}^{\text{II}} = E_{\text{crit}} - E_n^\lambda > E_{\text{crit}}$ for *repulsive* Coulomb potential which disappear for $\lambda \rightarrow 0$.

In [10] these results have been confirmed by a direct solution of the corresponding NC radial Schrödinger equation. The analytic properties of $S_j^\lambda(E)$ in the complex energy plane indicate that the aspects of causality in NCQM, within investigated model, are fully consistent with the standard rôle and interpretation of S-matrix poles.

In [14] we constructed the Laplace-Runge-Lenz vector $A_\lambda^k(E)$, $k = 1, 2, 3$, within our NC Coulomb model. We found that at fixed energy level $E < 0$ and $E > E_{\text{crit}}$ the model shows dynamical SO(4) symmetry with two sets of bound states for $E = E_{\lambda n}^{\text{I}}$ and $E = E_{\lambda n}^{\text{II}}$, whereas for the scattering regime $0 < E < E_{\text{crit}} = 2/\lambda^2$ the dynamical symmetry is SO(3,1), exactly as in the standard Coulomb problem.

In conclusion we claim that the investigated Coulomb NC QM model, although containing unexpected non-perturbative features, is fully consistent with the usual QM postulates and interpretation.

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