

EXAMPLES OF QUANTUM HOLONOMY WITH TOPOLOGY CHANGES

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ABSTRACT. We study a family of closed quantum graphs described by one singular vertex of order $n = 4$. By suitable choice of the parameters specifying the singular vertex, we can construct a closed path in the parameter space that physically corresponds to the smooth interpolation of different topologies - a ring, separate two lines, separate two rings, two rings with a contact point. We find that the spectrum of a quantum particle on this family of graphs shows quantum holonomy.

KEYWORDS: quantum graph, boundary condition, eigenvalue holonomy.

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1. INTRODUCTION

The idea of quantum mechanics on graphs has been introduced in the last century. A quantum graph is a metric graph with a Hamiltonian acting on functions on its edges. During the last 30 years, this concept has been developed into a powerful tool for the study of particles on tiny graph-like structures [5]. Since a graph is a one-dimensional variety, the analysis of a quantal particle on a graph consists in solving a system of *ordinary* differential equations, which makes the graph models simple from the mathematical point of view. Therefore, quantum mechanics on graphs can also serve as a useful laboratory for the study of various complex quantum phenomena in a simple setting.

Here we examine the influence of the topology of quantum graphs on spectral properties, taking inspiration from the works [1] and [7]. We may regard that the topology change mimics a macroscopic or violent variation of systems, e.g., the rupture and connection of quantum wires. We will show that a parametric evolution of an eigenenergy may form an open trajectory along a cycle involving the topology changes of graphs. If this is the case, the trajectory of the corresponding eigenspace is also open. Such a discontinuity of individual eigenobjects is called quantum holonomy [2, 4, 8, 10].

2. THE MODEL

Let us consider a node with four outgoing one-dimensional lines of finite length. Let the first two lines be stuck together at their outer endpoints to form a ring of length L_1 , and likewise, the second two lines be stuck together at their outer endpoints to form a ring of length L_2 , giving the whole object the shape of the character “∞” (Fig. 1). We assume there is no potential on the lines outside the central vertex. Therefore, a quantum particle confined to this object

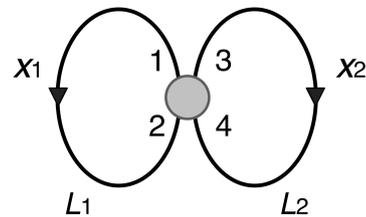


FIGURE 1. A closed quantum graph with an $n = 4$ singular vertex.

moves freely on the lines, and the only non-trivial part of the system is the connection condition of the wave function at the node.

We assign the coordinates $x_1 \in [0, L_1]$ and $x_2 \in [0, L_2]$ to the left ring and the right ring. The values $x_1 = 0$ and $x_2 = 0$ correspond to the inner endpoints 1 and 3, respectively, whereas the values L_1 and L_2 correspond to the inner endpoints 2 and 4, see Fig. 1. The behavior of a particle in the central vertex of degree 4 is determined by a boundary condition in the vertex. The general form of a boundary condition in a vertex of degree n follows from the study of self-adjoint extensions of the Laplacian operator on a graph. The boundary condition consists of n equations connecting the boundary values and derivatives that can be written in the form [6]

$$A\Psi + B\Psi' = 0, \quad (1)$$

$$AB^\dagger = BA^\dagger \quad \text{and} \quad \text{rank}(A|B) = n \quad (2)$$

with $(A|B)$ denoting the $n \times 2n$ matrix composed from the columns of matrices A and B . For the graph depicted in Figure 1, we have

$$\Psi = \begin{pmatrix} \psi_1(0) \\ \psi_1(L_1) \\ \psi_2(0) \\ \psi_2(L_2) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'_1(0) \\ -\psi'_1(L_1) \\ \psi'_2(0) \\ -\psi'_2(L_2) \end{pmatrix}. \quad (3)$$

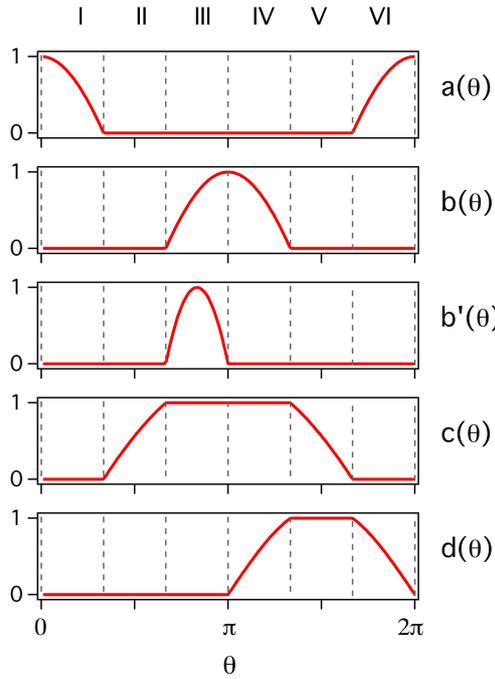


FIGURE 2. Graphs of functions defined by (7) for $\theta \in [0, 2\pi]$.

In our model, we focus on a specific subset of all possible connection conditions (1), namely

$$\begin{pmatrix} 1 & tb(\theta) & ta(\theta) & sd(\theta) \\ 0 & 1-c(\theta) & 0 & ta(\theta) \\ 0 & 0 & c(\theta) & tb'(\theta) \\ 0 & 0 & 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -tb(\theta) & c(\theta) & 0 & 0 \\ -ta(\theta) & 0 & 1-c(\theta) & 0 \\ -sd(\theta) & -ta(\theta) & -tb'(\theta) & 1 \end{pmatrix} \Psi \quad (4)$$

where t and s are positive constants, and $a(\theta)$, $b(\theta)$, $b'(\theta)$, $c(\theta)$ and $d(\theta)$ are real periodic functions of θ with period 2π that vary within the range between 0 and 1. In addition, it can easily be shown that the functions must satisfy

$$a(\theta) \neq 0 \implies c(\theta) = 0, \quad (5)$$

$$b(\theta), b'(\theta) \neq 0 \implies c(\theta) = 1 \quad (6)$$

for all $\theta \in [0, 2\pi]$ in order that the matrices A , B involved in the connection condition (4) obey (2). It can be shown that the boundary condition (4) never requires the wave function to obey $\psi_1(0) = \psi_1(L_1) = \psi_2(0) = \psi_2(L_2)$. This implies a generic discontinuity of the wave function in the central vertex, for which we call the vertex ‘‘singular’’.

2.1. LONG CYCLE

We first consider the boundary conditions (4) with functions $a(\theta)$, $b(\theta)$, $b'(\theta)$, $c(\theta)$ and $d(\theta)$ chosen as

$$a(\theta) = \cos \theta - \frac{1}{2} + \left| \cos \theta - \frac{1}{2} \right|,$$

$$\begin{aligned} b(\theta) &= -\cos \theta - \frac{1}{2} + \left| \cos \theta + \frac{1}{2} \right|, \\ b'(\theta) &= \frac{1}{4-2\sqrt{3}} \left\{ \sin \theta - \sqrt{3} \cos \theta - \sqrt{3} \right. \\ &\quad \left. - \left| \sin \theta - \sqrt{3} \cos \theta - \sqrt{3} \right| \right\}, \\ c(\theta) &= \frac{1}{\sqrt{3}-1} \left\{ \frac{\sqrt{3}}{2} - \frac{1}{2} + \left| \sin \frac{\theta}{2} - \frac{1}{2} \right| \right. \\ &\quad \left. - \left| \sin \frac{\theta}{2} - \frac{\sqrt{3}}{2} \right| \right\}, \\ d(\theta) &= \frac{1}{2\sqrt{3}} \left\{ \left| \sin \theta \right| - \sin \theta + \sqrt{3} \right. \\ &\quad \left. - \left| \left| \sin \theta \right| - \sin \theta - \sqrt{3} \right| \right\}. \quad (7) \end{aligned}$$

The graphs of these functions for $\theta \in [0, 2\pi]$ are plotted in Fig. 2. The interval $[0, 2\pi]$ can be divided into six regions, in each of which the vector $(a(\theta), b(\theta), b'(\theta), c(\theta), d(\theta))$ has a different structure of zero and nonzero entries. As we show below, these regions correspond to mutually different topologies of the system. This extends the basic idea of parametric systems with a varying topology introduced in the works of Balachandran *et. al.* [1] and also in [7]. Our use of a single $n = 4$ singular vertex makes the treatment more systematic and more unified.

Region I. For $\theta \in [0, \frac{\pi}{3}]$, we have

$$\begin{pmatrix} 1 & t \cdot a(\theta) & & \\ & 1 & & \\ & & t \cdot a(\theta) & \\ & & & t \cdot a(\theta) \end{pmatrix} \Psi' = \begin{pmatrix} & & & \\ -t \cdot a(\theta) & & & \\ & -t \cdot a(\theta) & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Psi, \quad (8)$$

where $ta(\theta)$ takes the value t at $\theta = 0$ and it continuously decreases down to 0 at $\theta = \frac{\pi}{3}$. With regard to the structure of (8), endpoint 1 is joined to endpoint 2 and endpoint 3 is joined to endpoint 4. On the other hand, endpoints 1 and 2 are separated, so are 3 and 4. Therefore, the system is topologically equivalent to a single ring, see Fig. 3.

Region II. For $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, we have

$$\begin{pmatrix} 1 & & & \\ & 1-c(\theta) & & \\ & & c(\theta) & \\ & & & c(\theta) \end{pmatrix} \Psi' = \begin{pmatrix} & & & \\ c(\theta) & & & \\ & 1-c(\theta) & & \\ & & & 1 \end{pmatrix} \Psi, \quad (9)$$

where we start with the value $c(\frac{\pi}{3}) = 0$ and continuously increase it up to $c(\frac{2\pi}{3}) = 1$. Since all four inner

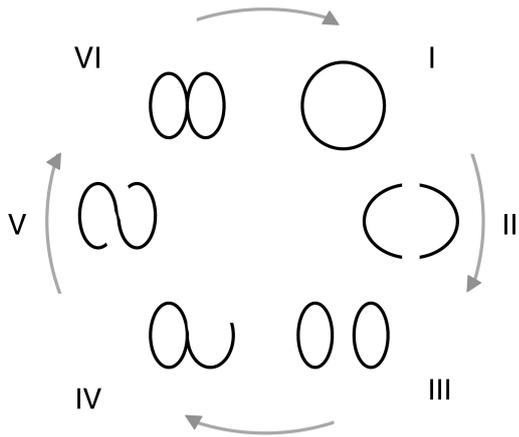


FIGURE 3. Cycle of topological changes corresponding to functions (7).

endpoints are separated, the system is topologically equivalent to two separate lines.

Region III. For $\theta \in [\frac{2\pi}{3}, \pi]$, we have

$$\begin{pmatrix} 1 & t \cdot b(\theta) & & \\ & 1 & t \cdot b'(\theta) & \\ & & & \end{pmatrix} \Psi' = \begin{pmatrix} -t \cdot b(\theta) & 1 & & \\ & & & \\ & & & \\ & & -t \cdot b'(\theta) & 1 \end{pmatrix} \Psi, \quad (10)$$

where $tb(\theta)$ continuously increases from $tb(\frac{2\pi}{3}) = 0$ up to $tb(\pi) = t$, while $tb'(\theta)$ takes the value $tb'(\theta) = 0$ at $\theta = \frac{2\pi}{3}$, continuously increases up to t at $\theta = \frac{5\pi}{6}$ and comes back down to 0 at $\theta = \pi$. The connection condition (10) implies separated endpoints 1 and 3, as well as 2 and 4, making the system topologically equivalent to two separate rings.

Region IV. For $\theta \in [\pi, \frac{4\pi}{3}]$, we have

$$\begin{pmatrix} 1 & t \cdot b(\theta) & & s \cdot d(\theta) \\ & 1 & & \\ & & & \\ & & & \end{pmatrix} \Psi' = \begin{pmatrix} -t \cdot b(\theta) & 1 & & \\ & & & \\ -s \cdot d(\theta) & & & 1 \end{pmatrix} \Psi, \quad (11)$$

where $tb(\theta)$ continuously decreases from $tb(\frac{\pi}{2}) = t$ to $tb(\frac{4\pi}{3}) = 0$, while $sd(\theta)$ continuously increases from $sd(\frac{\pi}{2}) = 0$ up to $sd(\frac{4\pi}{3}) = s$. The system is topologically equivalent to a ring and a line connected at a single point since three of four endpoints, namely 1, 2 and 4, are connected in the node. Note that at the juncture of regions III and IV, the system is topologically equivalent to a separated line and a ring.

Region V. For $\theta \in [\frac{4\pi}{3}, \frac{5\pi}{3}]$, we have

$$\begin{pmatrix} 1 & & s \\ & 1 - c(\theta) & \\ & & c(\theta) \end{pmatrix} \Psi' = \begin{pmatrix} & c(\theta) & \\ & & 1 - c(\theta) \\ -s & & 1 \end{pmatrix} \Psi, \quad (12)$$

where c continuously decreases from $c(\frac{4\pi}{3}) = 1$ down to $c(\frac{5\pi}{3}) = 0$. The system is topologically equivalent to a single line, since only endpoints 1 and 4 are connected, whereas 2 and 3 are separated.

Region VI. For $\theta \in [\frac{5\pi}{3}, 2\pi]$, we have

$$\begin{pmatrix} 1 & t \cdot a(\theta) & s \cdot d(\theta) \\ & 1 & t \cdot a(\theta) \\ & & & \end{pmatrix} \Psi' = \begin{pmatrix} & & & \\ -t \cdot a(\theta) & & & 1 \\ -s \cdot d(\theta) & -t \cdot a(\theta) & & 1 \end{pmatrix} \Psi, \quad (13)$$

where $ta(\theta)$ continuously increases from $ta(\frac{5\pi}{3})$ up to $ta(2\pi) = t$, while $sd(\theta)$ continuously decreases from $sd(\frac{5\pi}{3}) = s$ down to $sd(2\pi) = 0$. In this region, the system is topologically equivalent to two rings connected at a single point since all four endpoints are connected together at the central node.

Therefore, with the parameter change $\theta = 0 \rightarrow 2\pi$, seven distinct topologies are traversed in a continuous manner. The situation is illustrated in Fig. 3. The system is never identical for any two different values of θ in $[0, 2\pi)$, because the functions $a(\theta)$ to $d(\theta)$ form mutually different combinations of patterns in all six regions.

2.2. SHORT CYCLE

As an additional model, we consider a simplified parametric cycle represented by the boundary condition (4) with

$$\begin{aligned} a(\theta) &= \frac{1}{2} (\cos \theta + |\cos \theta|), \\ b(\theta) &= b'(\theta) = 0, \\ c(\theta) &= \frac{1}{2} (-\cos \theta + |\cos \theta|), \\ d(\theta) &= \frac{1}{2} (-\sin \theta + |\sin \theta|). \end{aligned} \quad (14)$$

The graphs of these functions are plotted in Fig. 4. This system traverses four distinct segments in the parameter space of θ , as θ runs from 0 to 2π . The segments correspond to four topologies, cf. Fig. 5. Namely, a ring, two disjoint lines, a line attached to a ring, and two rings connected at a point.

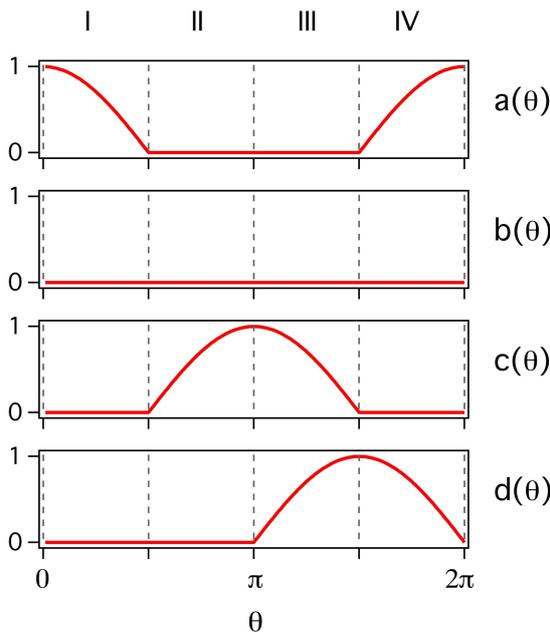


FIGURE 4. Graphs of functions defined by (14) for $\theta \in [0, 2\pi]$.

3. QUANTUM HOLONOMY

The system wave functions can be written in the form

$$\begin{aligned} \psi_1(x_1) &= \alpha_1 \sin kx_1 + \beta_1 \cos kx_1, \\ \psi_2(x_2) &= \alpha_2 \sin kx_2 + \beta_2 \cos kx_2. \end{aligned} \tag{15}$$

In this notation, we have

$$\Psi = U \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}, \quad \Psi' = kV \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} \tag{16}$$

with

$$U = \begin{pmatrix} & & & 1 \\ \sin kL_1 & \cos kL_1 & & \\ & & & 1 \\ & & \sin kL_2 & \cos kL_2 \end{pmatrix}, \tag{17}$$

$$V = \begin{pmatrix} & & & 1 \\ -\cos kL_1 & \sin kL_1 & & \\ & & & 1 \\ & & -\cos kL_2 & \sin kL_2 \end{pmatrix}.$$

The spectra of the system can be calculated from the secular equation

$$|AU + kB V| = 0. \tag{18}$$

It turns out that certain choices of s, t and ratio L_1/L_2 result in a quantum holonomy. Below we present the results, separately for the long cycle and the short cycle.

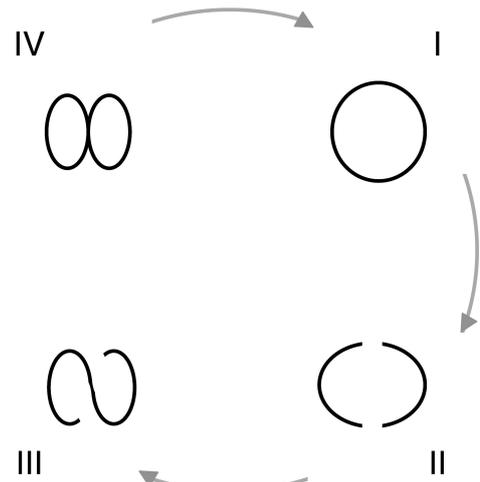


FIGURE 5. Cycle of topological changes corresponding to functions (14).

3.1. LONG CYCLE

We first consider the long cycle of parameter change represented by (7). We performed a numerical calculation of the spectral sets of the system for six different settings of the parameters $s, t, L_1/L_2$. We studied two combinations of s, t , namely

- $t = 0.1, s = 1,$
- $t = 0.5, s = 0.5,$

and for each of them, we considered three different ratios L_1/L_2 (with $L_1 + L_2 = 1$), namely

- the golden mean $L_1/L_2 = (1 + \sqrt{5})/2,$
- the silver mean $L_1/L_2 = 1 + \sqrt{2},$
- the bronze mean $L_1/L_2 = (3 + \sqrt{13})/2.$

We explain the numerical result shown in Figure 6. The spectral sets have period 2π as functions of θ . This is because all the Hamiltonians are 2π periodic by construction. On the other hand, however, some wavenumber do not have the 2π periodicity. For example, let us keep track of the wavenumber (say k_0) of the ground state at $\theta = 0$ in the uppermost figure. Around $\pi/3 < \theta < 2\pi/3$, k_0 increase suddenly and form three crossings with other eigenstates. Because the graphs consist of two disjoint parts, these crossings are not avoided. The eigenfunction corresponding to k_0 is localized within a disjoint fragment, and the other three eigenfunctions involving the crossings are localized in the other part. Hence, as a result of a 2π -cycle of θ , k_0 becomes the wavenumber of the third excited state. This is an example of the holonomy in eigenvalue and eigenspace. Some other wavenumbers, e.g., 4-th excited state at $\theta = 0$ in the leftmost figure, are not involved by the holonomy. By changing the ratio L_1/L_2 , we see various patterns of the quantum holonomy. Also, the holonomies appear in various choice of the parameters, cf. Figure 7.

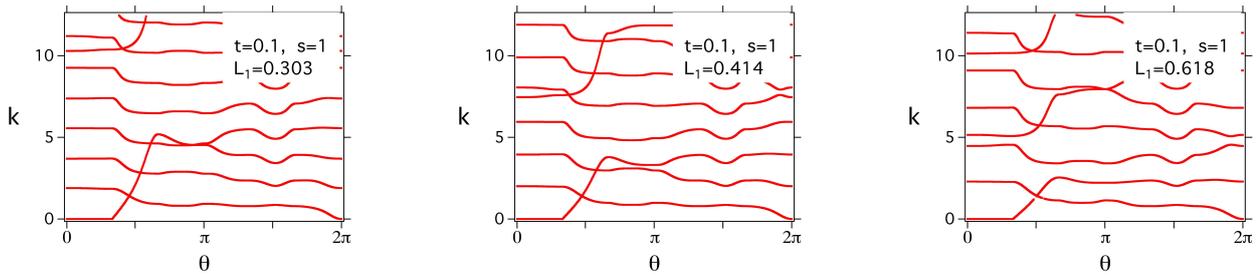


FIGURE 6. Holonomy as a result of the long cycle topology change for $t = 0.1, s = 1$. The ratio L_1/L_2 is chosen as the bronze mean (top), silver mean (middle) and golden mean (bottom).

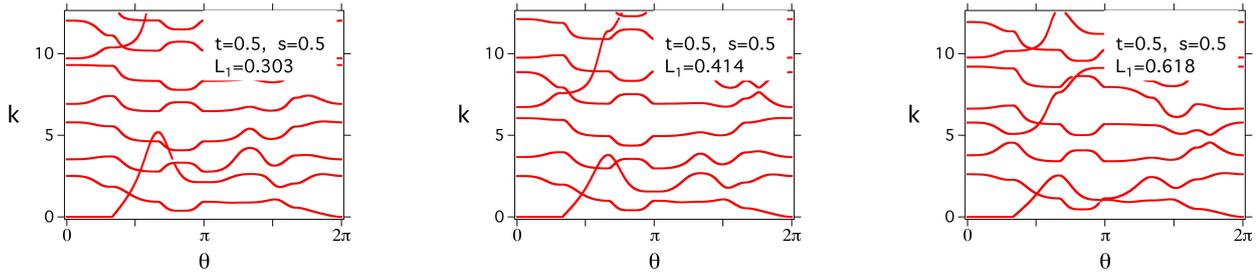


FIGURE 7. Holonomy as a result of the long cycle topology change for $t = 0.5, s = 0.5$. The ratio L_1/L_2 is chosen as the bronze mean (top), silver mean (middle) and golden mean (bottom).

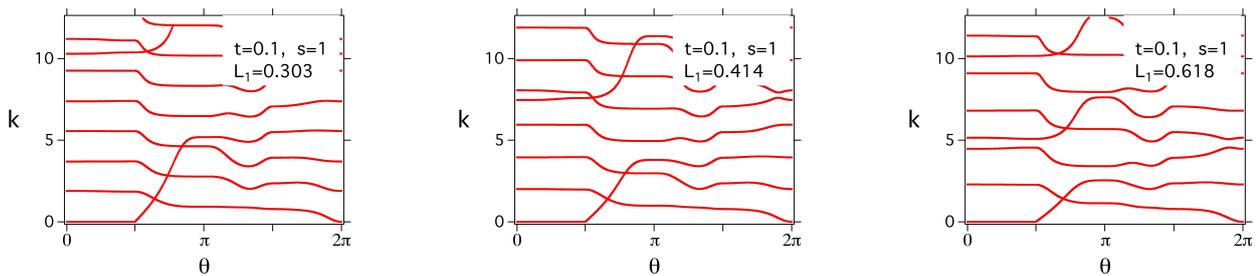


FIGURE 8. Holonomy as a result of the short cycle topology change for $t = 0.1, s = 1$. The ratio L_1/L_2 is chosen as the bronze mean (top), silver mean (middle) and golden mean (bottom).

3.2. SHORT CYCLE

Let us proceed to the simplified cycle represented by (14), in which four distinct topologies are connected in a single sequence. The spectra as functions of the angle parameter $\theta \in [0, 2\pi)$ have been calculated for the choice of parameters $t = 0.1, s = 1$ and L_1/L_2 attaining the value of the golden mean, the silver mean and the bronze mean. The results are depicted in Fig. 8.

4. ON THE BOUNDARY CONDITION

The boundary condition (1) represents a certain singular potential in the vertex that has in general a highly nontrivial nature. As has been shown in [3, 9], the physical content of a generic boundary condition corresponds to a set of infinitely strong δ potentials sited in points in an infinitely small web located in the vertex. We sketch what the webs look like for the boundary conditions of our model in Figures 9, 10 and 11. In all of the finite approximations, the parameter

ϵ has to be chosen small with respect to the wavelength of the particle. For $\epsilon \rightarrow 0$, the approximation effectively produces the boundary conditions (4). The result has been obtained using the formulas from [3, 9]. It illustrates in a simple way the relation between the global topology of the system and the topology of the internal infinitely small web.

5. DISCUSSION

The creation of the holonomy in our model can be explained in the following way. In regions I, IV, V and VI, when the graph consists of a single component, the eigenvalues as functions of the system parameter θ can change their values, but they never cross each other, because crossings will be avoided generically. However, in regions II and III, the graph is separated into two components (two lines or two rings), i.e., the system is in fact made up of two independent entities, both having its own energy levels that can independently vary with change of the system param-

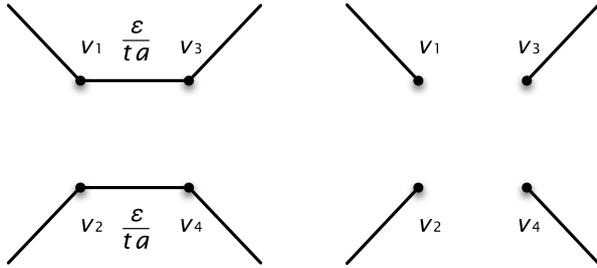


FIGURE 9. Finite approximation of the physical meaning of boundary condition (4) for θ belonging to sectors I and II. The strengths of the δ potentials in vertices 1, 2, 3, 4 are given by the following formulas. Left: $v_1 = v_2 = \frac{(ta)^2 - ta}{\epsilon}$, $v_3 = v_4 = \frac{1 - ta}{\epsilon}$. Right: $v_1 = 0$, $v_2 = \frac{c}{1 - c}$, $v_3 = \frac{1 - c}{c}$, $v_4 = \frac{1}{\epsilon}$.

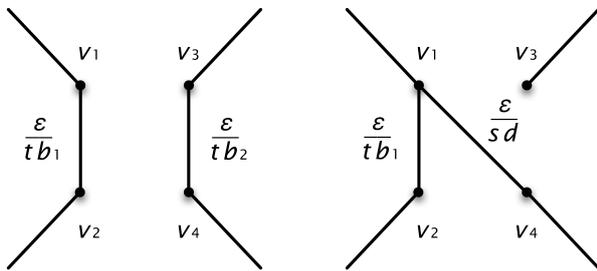


FIGURE 10. Finite approximation of the physical meaning of boundary condition (4) for θ belonging to sectors III and IV. The strengths of the δ potentials in vertices 1, 2, 3, 4 are given by the following formulas. Left: $v_1 = \frac{(tb)^2 - tb}{\epsilon}$, $v_2 = \frac{1 - tb}{\epsilon}$, $v_3 = \frac{(tb')^2 - tb'}{\epsilon}$, $v_4 = \frac{1 - tb'}{\epsilon}$. Right: $v_1 = \frac{(tb_1)^2 + (sd)^2 - tb_1 - sd}{\epsilon}$, $v_2 = \frac{1 - tb_1}{\epsilon}$, $v_3 = \frac{1}{\epsilon}$, $v_4 = \frac{1 - sd}{\epsilon}$.

eter θ . Therefore, if the parametrization of boundary conditions is conveniently chosen, the eigenvalues as functions of θ can cross each other, which leads to the holonomy effect.

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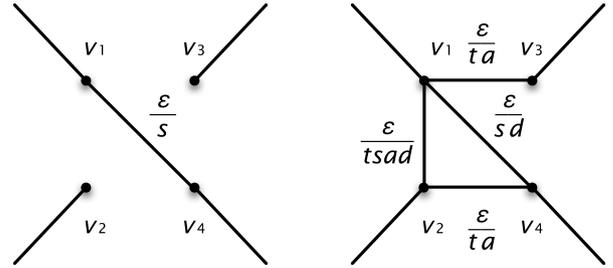


FIGURE 11. Finite approximation of the physical meaning of the boundary condition (4) for θ belonging to the sectors V and VI. The strengths of the δ potentials in the vertices 1,2,3,4 are given by the following formulas. Left: $v_1 = \frac{s^2 - s}{\epsilon}$, $v_2 = \frac{c}{1 - c}$, $v_3 = \frac{1 - c}{c}$, $v_4 = \frac{1 - s}{\epsilon}$. Right: $v_1 = \frac{(ta)^2 + (sd)^2 - ta - sd - 3tsad}{\epsilon}$, $v_2 = \frac{(ta)^2 - ta - 3tsad}{\epsilon}$, $v_3 = \frac{1 - ta}{\epsilon}$, $v_4 = \frac{1 - ta - sd}{\epsilon}$.

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