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MAXIMAL SUBSETS OF PAIRWISE SUMMABLE ELEMENTS IN GENERALIZED EFFECT ALGEBRAS

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Abstract. We show that in any generalized effect algebra $(G;\oplus,0)$ a maximal pairwise summable subset is a sub-generalized effect algebra of $(G;\oplus,0)$, called a summability block. If G is lattice ordered, then every summability block in G is a generalized MV-effect algebra. Moreover, if every element of G has an infinite isotropic index, then G is covered by its summability blocks, which are generalized MV-effect algebras in the case that G is lattice ordered. We also present the relations between summability blocks and compatibility blocks of G. Counterexamples, to obtain the required contradictions in some cases, are given.

Keywords: (generalized) effect algebra, MV-effect algebra, summability block, compatibility block, linear operators in Hilbert spaces.

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1. Introduction and some basic definitions

In a Hilbert space formalization of quantum mechanics, G. Birkhoff and J. von Neumann proposed the concept of quantum logics (in 1936 the concept of modular ortholattices and later orthomodular lattices, discovered by Husimi in 1937). Nevertheless, in the set $\mathcal{P}(\mathcal{H})$ of all projection operators in a separable Hilbert space (used as a model for orthomodular lattices) every event satisfies the non-contradiction principle. Thus the set $\mathcal{P}(\mathcal{H})$ is not the set of all possible events in quantum theory. In 1994, D. Foulis introduced algebraic structures called effect algebras. Equivalent structures, in some sense, are D-posets introduced by Kôpka and Chovanec in 1994. The prototype for the axiomatic system of effect algebras was the set $\mathcal{E}(\mathcal{H})$ of all positive linear operators dominated by the identity operator in a Hilbert space. Events in $\mathcal{E}(\mathcal{H})$, called effects, do not satisfy the non-contradiction law (meaning that there exist unsharp events x and non x which are not disjoint). They represent unsharp measurements or observations on a quantum mechanical system in a Hilbert space $\mathcal{H}$. Moreover, a special kind of effect algebras are MV-algebras, which are algebraic bases for multivalued logic, as a generalization of Boolean algebras. Effect algebras are very suitable algebraic structures for being carriers of probability measures when events may be unsharp or pairwise non-compatible.

The mutually equivalent generalizations (unbounded version) of effect algebras were introduced in 1994 by several authors — D. Foulis and M.K. Bennett, G. Kalmbach and Z. Riečanová, J. Hedlíková and S. Pulmannová, and F. Kôpka and F. Chovanec. On the other hand, all intervals in these generalized effect algebras are effect algebras.

Recently, operator representations of abstract effect algebras (i.e. their isomorphism with sub-effect algebras of the standard effect algebra $\mathcal{E}(\mathcal{H})$ mentioned above) have been studied. It was proved in [14] that the set $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ of all positive linear operators in an infinite-dimensional complex Hilbert space $\mathcal{H}$ with partially defined sum of operators (which coincides with the usual sum) restricted to the common domains of operators forms a generalized effect algebra. This generalized effect algebra $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ is a union of sub-generalized effect algebras of maximal subsets of pairwise summable operators. Moreover, all intervals are effect algebras isomorphic to sub-effect algebras of the standard effect algebra $\mathcal{E}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (see [13]).

We are going to show that in a generalized effect algebra G without elements with finite isotropic indexes (which corresponds to the operator case) its maximal subsets of pairwise summable elements are sub-generalized effect algebras. Moreover, such G is covered by those sub-generalized effect algebras.

Definition 1 ([3]). A partial algebra $(E;\oplus,0,1)$ is called an *effect algebra* if $0,1\in E$ are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $x,y,z\in E$:

(Ei) $x\oplus y=y\oplus x$ if $x\oplus y$ is defined,

(Eii) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$ if one side is defined, (Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

The basic references for the present text are the books by Dvurečenskij and Pulmannová [2], and Blank, Exner and Havlíček [1], where unexplained terms and notations concerning the subject can be found.

In 1994 also a generalization of effect algebras without a top element was introduced by several authors ([3, 5, 6, 8]).

Definition 2. A partial algebra $(E; \oplus, 0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $x, y, z \in E$:

(GEi) $x \oplus y = y \oplus x$, if one side is defined,

(GEii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, if one side is defined,

(GEiii) $x \oplus 0 = x$,

(GEiv) $x \oplus y = x \oplus z$ implies $y = z$ (cancellation law),

(GEv) $x \oplus y = 0$ implies $x = y = 0$.

In every (generalized) effect algebra E relation $\leq$ and the partial binary operation $\ominus$ can be defined by

(PO) $x \leq y$ iff there exists $z \in E$ such that $x \oplus z = y$. In that case, such element z is unique and we set $z = y \ominus x$.

Then $\leq$ is a partial order on E under which 0 is the least element of E.

A generalized effect algebra $(E; \oplus, 0)$ is called a *lattice generalized effect algebra* if E with respect to induced partial order $\leq$ is a lattice.

Definition 3. Let $(E; \oplus, 0, 1)$ be an effect algebra ($(E; \oplus, 0)$ be a generalized effect algebra). A subset $Q \subseteq E$ is called a *sub-effect algebra* (*sub-generalized effect algebra*) of E iff

(Si) $1 \in Q$ ($0 \in Q$),

(Sii) if $a, b, c \in Q$ with $a \oplus b = c$ and out of a, b, c at least two elements are in Q then $a, b, c \in Q$.

Let $(E; \oplus, 0, 1)$ be an effect algebra and $F \subseteq E$, by the symbol $\oplus_{/F}$ we will denote a restriction of $\oplus$ to F, i.e. for $a, b \in F$, $a \oplus_{/F} b$ is defined if and only if $a \oplus b$ is defined and $a \oplus_{/F} b = a \oplus b$.

It is easy to see that sub-effect algebra (sub-generalized effect algebra) Q of $(E; \oplus, 0, 1)$ ($(E; \oplus, 0)$) is an effect algebra $(Q; \oplus_{/Q}, 0, 1)$ (generalized effect algebra $(Q; \oplus_{/Q}, 0)$) in its own right.

Definition 4. Let $(E; \oplus, 0)$ be a generalized effect algebra. For any $x \in E$, if there exists a natural number $ord(x) \in \mathbb{N}$ such that $ord(x) \cdot x = x \oplus x \oplus \ldots \oplus x$ ($ord(x)$-times) is defined, but $(ord(x) + 1) \cdot x$ is not defined, is called an *isotropic index of x*. If such natural number does not exist, we set $ord(x) = \infty$.

Definition 5. Elements $a, b \in E$ of an effect algebra $(E; \oplus, 0, 1)$ (generalized effect algebra $(E; \oplus, 0)$) are called *compatible* (we write $a \leftrightarrow b$) if there exist $a_1, c, b_1 \in E$ such that $a_1 \oplus c \oplus b_1$ is defined and $a = a_1 \oplus c$, $b = b_1 \oplus c$.

In [11] it was proved that in any lattice effect algebra E for $a, b \in E$ we have $a \leftrightarrow b$ iff $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ is defined in E. Moreover, we call every maximal subset of pairwise compatible elements of E a *compatibility block* of E. Every lattice effect algebra E is a set-theoretical union of its compatibility blocks [10, Theorem 3.2]. A lattice effect algebra possessing a unique block is called an *MV-effect algebra* (hence $a \leftrightarrow b$ for all $a, b \in E$).

2. Pairwise summable generalized effect algebras

Recall that elements a, b of a generalized effect algebra $(G; \oplus, 0)$ are called *summable* if $a \oplus b$ exists in G.

A nonempty subset F of a generalized effect algebra $(G; \oplus, 0)$ is called a *pairwise summable subset* of G if $a \oplus b$ exists for every not necessarily different elements $a, b \in F$ and $a \oplus b \in F$ (hence F is closed under the partial operation $\oplus$). Evidently, in this case, every $a \in F$ has the infinite isotropic index $ord(a) = \infty$.

A (sub-) generalized effect algebra E of $(G; \oplus, 0)$ is called *pairwise summable* if it is a pairwise summable subset of G.

We are going to show that every maximal subset of pairwise summable elements of a generalized effect algebra G is a sub-generalized effect algebra of G. Moreover, we study further properties of these pairwise summable sub-generalized effect algebras.

Theorem 1. *Let $(G; \oplus, 0)$ be a generalized effect algebra. Let a non-empty subset F of G satisfy the following conditions:*

(1.) For every $a, b \in F$ there exists $a \oplus b \in F$,

(2.) If $a \in G$ and $a \oplus e$ exists for every $e \in F$ then $a \in F$.

Then F is a sub-generalized effect algebra of G.

Proof. By (2.) we obtain $0 \in F$, since $0 \oplus e$ exists for all $e \in F$.

Suppose now that $a \oplus b = c$, for $a, b, c \in G$. If $a, b \in F$ then $c = a \oplus b \in F$ by (1.). Further, in the case $a, c \in F$, we have $b = c \ominus a \leq c$. Since $c \oplus e$ exists for all $e \in F$, also $b = (c \ominus a) \oplus e$ exists for all $e \in F$. Thus by (2.) $b = c \ominus a \in F$. That is, F is a sub-generalized effect algebra of $(G; \oplus, 0)$. ■

Remark 1. Condition (1.) on a subset $F \subseteq G$ of a generalized effect algebra $(G; \oplus, 0)$ in the above theorem guarantees that F is a pairwise summable subset of G. Condition (2.) then provides that F is a maximal pairwise summable subset of G.

Definition 6. Let $(G;\oplus,0)$ be a generalized effect algebra and $F\subseteq G$ be a subset of G that satisfies conditions (1.) and (2.) from Theorem 1. Then F is called a *summability block* of G.

Corollary 1. *Every maximal pairwise summable subset $F\subseteq G$ (a summability block) of elements of any generalized effect algebra $(G;\oplus,0)$ is a sub-generalized effect algebra of G.*

Example 1. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and

$$\mathcal{D}=\{D\subseteq\mathcal{H}\mid D \text{ is a dense sub-space of } \mathcal{H}\}.$$

Let $\mathcal{V}_{\mathcal{D}}(\mathcal{H})=$

$$\bigl\{A:D(A)\to\mathcal{H}\;\big|\;(Ax,x)\geq 0 \text{ for all } x\in D(A),\\ D(A)\in\mathcal{D},\, D(A)=\mathcal{H} \text{ if } A \text{ is bounded}\bigr\}$$

be a set of densely defined positive linear operators on $\mathcal{H}$. In [14] it was shown that $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ with the partial binary operation $\oplus_{\mathcal{D}}$ defined for every $A,B\in\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ by $A\oplus_{\mathcal{D}}B=A+B$ (the usual sum) if A or B is bounded or $D(A)=D(B)$ if A,B are both unbounded, forms a generalized effect algebra $(\mathcal{V}_{\mathcal{D}}(\mathcal{H});\oplus_{\mathcal{D}},\mathbf{0})$. Moreover, for every $D\in\mathcal{D}$ the set

$$\mathcal{G}_D(\mathcal{H})=\bigl\{\mathcal{A}\in\mathcal{V}_{\mathcal{D}}(\mathcal{H})\;\big|\;\mathcal{A} \text{ is bounded},\\ \text{or } D(A)=D\bigr\}$$

is a sub-generalized effect algebra of $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ (see [14]).

For every $A,B\in\mathcal{G}_D(\mathcal{H})$, $D\in\mathcal{D}$ by definition of $\oplus$ the condition (1.) is satisfied. Let us assume that there exists $C\in\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ such that $C\oplus A$ is defined for all $A\in\mathcal{G}_D(\mathcal{H})$. Further, there exists some $B\in\mathcal{G}_D(\mathcal{H})$ with $D(B)=D\neq\mathcal{H}$ (if not, then $D=\mathcal{H}$), hence by the Hellinger-Toeplitz theorem B is unbounded. Since $C\oplus B$ is defined we have $D(C)=D(B)$, that is $C\in\mathcal{G}_D(\mathcal{H})$. Therefore sets $\mathcal{G}_D(\mathcal{H})$ are for $D\neq\mathcal{H}$ maximal pairwise summable sub-generalized effect algebras.

Example 2. According to [1], every positive linear operator $A\in\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ uniquely determines a positive sesquilinear form t_A on $D(t_A)=D(A)$ by $t_A(x,y)=(Ax,y)$. Let us denote a set of all such sesquilinear forms by $\mathcal{F}_{\mathcal{D}}(\mathcal{H})$, namely

$$\mathcal{F}_{\mathcal{D}}(\mathcal{H})=\bigl\{t:D(t)\times D(t)\to\mathcal{H}\;\big|\\ \text{there exists } A\in\mathcal{V}_{\mathcal{D}}(\mathcal{H}) \text{ with } D(A)=D(t)\\ \text{and } t(x,y)=(Ax,y) \text{ for all } x,y\in D(t)\bigr\}.$$

On the set $\mathcal{F}_{\mathcal{D}}(\mathcal{H})$, we can define a partial sum $t\oplus s$ for any $t,s\in\mathcal{F}_{\mathcal{D}}(\mathcal{H})$ in the following way: $t\oplus s$ exists whenever $D(t)=D(s)$ or t or s is bounded (then $D(t\oplus s)=D(t)\cap D(s)$) by $(t\oplus s)(x,y)=t(x,y)+s(x,y)$ for all $x,y\in D(t)\cap D(s)$. It is easy to show that $(\mathcal{F}_{\mathcal{D}}(\mathcal{H});\oplus,0)$ is a generalized effect algebra isomorphic to $(\mathcal{V}_{\mathcal{D}}(\mathcal{H});\oplus_{\mathcal{D}},\mathbf{0})$.

As in the previous example, maximal pairwise summable subsets are

$$\mathcal{M}_D(\mathcal{H})=\bigl\{t\in\mathcal{F}(H)\;\big|\;t \text{ is bounded, or } D(t)=D\bigr\},$$

hence they are sub-generalized effect algebras of $\mathcal{F}_{\mathcal{D}}(\mathcal{H})$.

Example 3. Let us consider Chang's effect algebra $(E;\oplus,0,1)$ which is defined by $E=\{0,a,2a,\dots,(2a)',a',1\}$. Consider its subset $F=\{0,a,2a,\dots\}\subseteq G$. Clearly F satisfies condition (1.). Since any element of the form $(n_0a)'$ is summable only with elements na for $n\leq n_0$, hence $(n_0a)'\notin F$ which gives that (2.) is satisfied as well.

3. Intervals in pairwise summable generalized effect algebras

The significant property of any generalized effect algebra $(G;\oplus,0,q)$ is the fact that for every non-zero element $q\in G$, the interval

$$[0,q]_G=\{a\in G\mid \text{there exists } c\in G \text{ with } a\oplus c=q\}$$

is an effect algebra $([0,q]_G;\oplus_q,0)$. The partial operation $\oplus_q$ is defined by $a\oplus_q b$ exists iff $a\oplus b\leq q$ and then $a\oplus_q b=a\oplus b$. Further, let us investigate intervals in pairwise summable generalized effect algebras. Namely, we are going to show that if G with derived $\leq$ is a lattice, then these intervals are MV-effect algebras (hence can be organized into MV-algebras). We start with the observation that every pairwise summable generalized effect algebra $(G;\oplus,0)$ is a generalized MV-effect algebra if and only if $(G,\leq)$ is a lattice.

Recall that a non-void subset $I\subseteq L$ of a partially ordered set $(L,\leq)$ is an *order ideal* if $a\in L, b\in I$ and $a\leq b$ implies $a\in I$.

Let $(P;\leq,0)$ be a generalized effect algebra. Let P^* be a set disjoint from P with the same cardinality. Consider a bijection $a\to a^*$ from P onto P^* and let us denote $P\,\dot{\cup}\,P^*$ by E. Further define a partial binary operation $\oplus^*$ on E by the following rules. For $a,b\in P$

(1.) $a\oplus^* b$ is defined if and only if $a\oplus b$ is defined, and $a\oplus^* b=a\oplus b$,

(2.) $b^*\oplus^* a$ and $a\oplus^* b^*$ are defined if and only if $b\ominus a$ is defined and then $b^*\oplus^* a=(b\ominus a)^*=a\oplus^* b^*$.

Theorem 2 ([2, p. 18]). *For every generalized effect algebra P and $E=P\,\dot{\cup}\,P^*$ the structure $(E;\oplus^*,0,0^*)$ is an effect algebra. Moreover, P is a proper order ideal in E closed under $\oplus^*$ and the partial order induced by $\oplus^*$, when restricted to P, coincides with the partial order induced by $\oplus$. The generalized effect algebra P is a sub-generalized effect algebra of E and for every $a\in P, a\oplus a^*=0^*$.*

Since the definition of $\oplus^*$ on $E=P\,\dot{\cup}\,P^*$ coincides with the $\oplus$-operation on P, it will cause no confusion

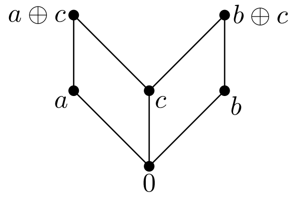

FIGURE 1. To Example 4: $(G; \oplus, 0)$

if from now on we use the notation $\oplus$ also for its extension to E.

To avoid undue repetitions on *generalized MV-effect algebras*, we recall the following statements from [11], giving their equivalent definitions.

Theorem 3 ([11, Theorem 3.2]). *For a generalized effect algebra P the following conditions are equivalent:*

(1.) *P is a generalized MV-effect algebra,*

(2.) *$E = P \dot{\cup} P^*$ is an MV-effect algebra.*

Theorem 4 ([11, Theorem 3.3]). *A generalized effect algebra P is a generalized MV-effect algebra iff the following conditions are satisfied*

(1.) *P is a lattice,*

(2.) *for all $a, b, c \in P$ the existence of $a \oplus c$ and $b \oplus c$ implies the existence of $(a \vee_P b) \oplus c$,*

(3.) *$\bigvee \{c \in P \mid a \oplus c$ exists and $c \leq b\}$ exists in P, for all $a, b \in P$,*

(4.) *$(a \ominus (a \wedge_P b)) \oplus (b \ominus (a \wedge_P b))$ exists for all $a, b \in P$.*

Lemma 1. *Let $(G; \oplus, 0)$ be a generalized MV-effect algebra. Then for every $q \in G$ the interval $[0, q]_G \subseteq G$ is an MV-effect algebra.*

Proof. Clearly, for every $q \in G$, the interval $[0, q]_G \subseteq G$ is lattice ordered, as for every $a, b \in [0, q]_G$ we have $a \vee_G b, a \wedge_G b \leq q$ in G.

Moreover, for every $a, b \in [0, q]_G \subseteq G$ we have $(a \ominus (a \wedge_G b)) \oplus (b \ominus (a \wedge_G b))$ exists in G by Theorem 4. Since $(a \ominus (a \wedge_G b)) \oplus (b \ominus (a \wedge_G b)) = [(a \ominus (a \wedge_G b)) \vee_G (b \ominus (a \wedge_G b))] \oplus [(a \ominus (a \wedge_G b)) \wedge_G (b \ominus (a \wedge_G b))] = (a \ominus (a \wedge_G b)) \vee_G (b \ominus (a \wedge_G b)) \leq a \vee_G b \leq q$ we have that $(a \ominus_q (a \wedge_q b)) \oplus_q (b \ominus_q (a \wedge_q b))$ also exists in $[0, q]_G$ (for inequalities see [2, p. 70]). This proves that a, b are compatible elements of a lattice effect algebra $([0, q]_G; \oplus_q, 0, q)$. Hence $([0, q]_G; \oplus_q, 0, q)$ is an MV-effect algebra. ∎

The converse of this lemma, in general, does not hold as can be seen in the following example.

Example 4. Let us have a generalized effect algebra $(G; \oplus, 0)$ given by $G = \{0, a, b, a \oplus c, b \oplus c\}$ (Fig. 1). Consider $E = G \dot{\cup} G^*$ (Fig. 2).

(1.) Clearly, a is not compatible with b. This is because $a \leftrightarrow b$ if and only if there exists $(a \ominus (a \wedge_G b)) \oplus (b \ominus (a \wedge_G b)) = a \oplus b$ which is not defined.

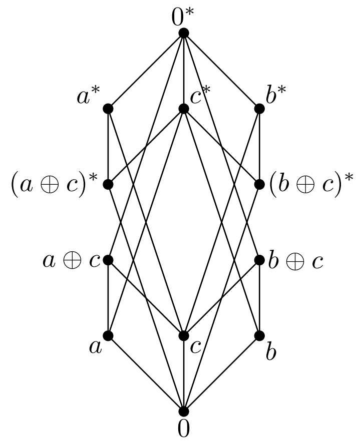

FIGURE 2. To Example 4: $E = G \dot{\cup} G^*$

(2.) $E = G \dot{\cup} G^*$ is a lattice.

(3.) G is a prelattice generalized effect algebra but it is not a generalized MV-effect algebra, since $E = G \dot{\cup} G^*$ is not an MV-effect algebra.

(4.) Every interval of G is an MV-effect algebra, namely $[0, a \oplus c]$, $[0, b \oplus c]$ are Boolean algebras which are MV-effect algebras, and $[0, a]$, $[0, b]$ and $[0, c]$ are finite chains, which are MV-effect algebras as well. Nevertheless, G is not a generalized MV-effect algebra since $E = G \dot{\cup} G^*$ is not an MV-effect algebra.

Using previous theorems we obtain statements for pairwise summable lattice ordered generalized effect algebras.

Theorem 5. *Let $(G; \oplus, 0)$ be a pairwise summable lattice ordered generalized effect algebra. Then*

(1.) *$(G; \oplus, 0)$ is a generalized MV-effect algebra,*

(2.) *$E = G \dot{\cup} G^*$ is an MV-effect algebra,*

(3.) *For every $q \in G$ the interval $[0, q]_G \subseteq G$ is an MV-effect algebra.*

Proof. Since G with derived partial order $\leq$ is a lattice and every pair of elements of G is summable, G satisfies all conditions (1.)–(4.) of Theorem 4. Further (2.) follows by Theorem 3 and (3.) by (1.) and Lemma 1. ∎

4. BLOCKS OF PAIRWISE SUMMABLE ELEMENTS IN GENERALIZED EFFECT ALGEBRAS

In [10] it was shown that every lattice effect algebra is a set theoretical union of blocks of compatible elements. In this section we present an analogous statement for blocks of pairwise summable elements in generalized effect algebras.

Let $(P, \leq)$ be a poset (e.g. generalized effect algebra). We call $(P, \leq)$ *inductive* if every chain in P has an upper bound.

Theorem 6. *Let $(G;\oplus,0)$ be a generalized effect algebra such that for every element $a \in G$, its isotropic index $ord(a) = \infty$. Then G is a set-theoretical union of its summability blocks, which are sub-generalized effect algebras of G.*

Proof. Let $A \subseteq G$ be a non-empty set of pairwise summable elements (i.e., $a \oplus b$ exists for every $a, b \in A$) and let $\mathcal{A} = \{B \subseteq G \mid A \subseteq B,\ B$ is a set of pairwise summable elements$\}$. Then for every chain $\mathcal{B} \subseteq \mathcal{A}$ (i.e., for $X, Y \in \mathcal{B}$ we have either $X \subseteq Y$ or $Y \subseteq X$) we show that $\bigcup \mathcal{B} \in \mathcal{A}$. Let us have $x, y \in \bigcup \mathcal{B}$. Then there exist B_x, B_y such that $x \in B_x, y \in B_y$. By assumptions $B_x \subseteq B_y$ or $B_y \subseteq B_x$, hence $x, y \in B_x$ or $x, y \in B_y$ i.e. $x \oplus y$ exists.

Therefore $\bigcup \mathcal{B} \in \mathcal{A}$ hence $\mathcal{A}$ is inductive. Thus a maximal element M by Zorn's Lemma exists in $\mathcal{A}$ and M is clearly a summability block of G.

For every $a \in G, a \neq 0$ there exists a pairwise summable subset $A = \{0, a, 2a, \dots\} \in G$. By previous a subset $A \subseteq G$ is contained in some summability block M. Thus G is a set-theoretical union of summability blocks M. Blocks are sub-generalized effect algebras by Theorem 1 (resp. Corollary 1). ■

Hence any generalized effect algebra without elements with finite isotropic index $(G;\oplus,0)$ is covered by its summability blocks.

Example 5. We now turn to Example 1. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Consider a generalized effect algebra $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$ and its sub-generalized effect algebras $\mathcal{G}_D(\mathcal{H})$ from Example 1. Then for any $D \in \mathcal{D}$, $D \neq \mathcal{H}$, $\mathcal{G}_D(\mathcal{H})$ forms a summability block of $\mathcal{V}_{\mathcal{D}}(\mathcal{H})$. Note that sub-generalized effect algebras $\mathcal{G}_D(\mathcal{H})$ are also compatibility blocks (see [14], hence in this case, compatibility and summability blocks coincide.

Theorem 7. *Let $(G;\oplus,0)$ be a lattice generalized effect algebra such that for every element $a \in G$, its isotropic index $ord(a) = \infty$. Then G is a set-theoretical union of its blocks, which are sub-generalized effect algebras of G and generalized MV-effect algebras in its own right.*

Proof. This follows from Theorems 5 and 6. ■

Corollary 2. *For every maximal pairwise summable subset F of a lattice ordered generalized effect algebra $(G;\oplus,0)$ and any $q \in G, q \neq 0$ the intervals $[0,q]_F = [0,q]_G \cap F$ are MV-effect algebras in its own right.*

Proof. The identical mapping $\varphi : [0,q]_G \to [0,q]_G$ restricted to $[0,q]_F = [0,q]_G \cap F$ is an embedding of $[0,q]_F$ into $[0,q]_G$. Thus if G is a lattice ordered generalized effect algebra, then $[0,q]_G \cap F$ is a sub-effect algebra of $[0,q]_G$ (see [7, Section 3]) and consequently it is an MV-effect algebra in its own right. ■

Example 6. Consider Chang's effect algebra $(G;\oplus,0)$ mentioned in Example 3. It is not covered by summability blocks since it has elements with finite isotropic index. There exists only one summability block $F = \{0, a, 2a, \dots\} \subseteq G$. On the other hand, since G is linearly ordered by induced partial order $\leq$, all of its elements are pairwise compatible, hence the only compatibility block is G itself (G is an MV-effect algebra). That is compatibility and summability blocks need not coincide.

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References

[1] Blank J., Exner P., Havlíček M., *Hilbert Space Operators in Quantum Physics*, 2nd ed. Springer, Berlin (2008).

[2] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht/Ister Science, Bratislava, 2000.

[3] Foulis D. J., Bennett M. K., *Effect Algebras and Unsharp Quantum Logics*, Found. Phys. **24** (1994), 1331–1352.

[4] Gudder S., *D-algebras*, Found. Phys. **26**, no. 6, (1996), 813–822.

[5] Hedlíková J., Pulmannová S., *Generalized Difference Posets and Orthoalgebras*, Acta Math. Univ. Comenianae LXV, (1996), 247–279.

[6] Kalmbach G., Riečanová Z. *An Axiomatization for Abelian Relative Inverses*, Demonstratio Math. **27**, (1994), 769–780.

[7] Janda J., Riečanová Z. *Intervals in Generalized Effect Algebras*, to appear in Soft Computing, (2013), doi:10.1007/s00500-013-1083-x.

[8] Kôpka F., Chovanec F. *D-posets*, Math. Slovaca **44**, (1994), 21–34.

[9] Riečanová Z., *Subalgebras, Intervals and Central Elements of Generalized Effect Algebras*, Inter. J. Theor. Phys. **38**, (1999), 3209–3220.

[10] Riečanová Z., *Generalization of Blocks for D-Lattices and Lattice-Ordered Effect Algebras*, Inter. J. Theor. Phys. **39**, (2000), No. 2., pp. 231–237.

[11] Riečanová Z., Marinová I. *Generalized Homogeneous, Prelattice and MV-Effect Algebras*, Kybernetika **41**, (2005), No. 2, pp. 129–142.

[12] Riečanová Z., Zajac M., *Intervals in Generalized Effect Algebras and their Sub-Generalized Effect Algebras*, Acta Polytechnica **53**, (2013), No. 3, pp. 314–316.

[13] Riečanová Z, Zajac M., *Hilbert Space Effect-Representations of Effect Algebras*, Rep. Math. Phys. **70**, (2012), No. 2, pp. 283–290.

[14] Riečanová Z., Zajac M., Pulmannová S., *Effect Algebras of Positive Linear Operators Densely Defined on Hilbert Spaces*, Rep. Math. Phys. **68**, (2011), 261–270.