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EIGENVALUE COLLISION FOR PT-SYMMETRIC WAVEGUIDE

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Abstract. We consider a model of a planar $\mathcal{PT}$-symmetric waveguide and study the phenomenon of the eigenvalue collision under perturbation of the boundary conditions. This phenomenon was discovered numerically in previous works. The main result of this work is an analytic explanation of this phenomenon.

Keywords: PT-symmetric operator, eigenvalues, perturbation, asymptotics.

1. Introduction and main results

In this paper we study a problem in the theory of $\mathcal{PT}$-symmetric operators which has been studied rather intensively after the pioneering works [12–21]. Our model is introduced as follows.

Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$, let Ω be the strip $\{x : -d < x_2 < d\}$, $d > 0$, and let $\alpha = \alpha(x_1)$ be a function in $W^1_\infty(\mathbb{R})$. We consider the operator $\mathcal{H}_\alpha$ in $L_2(\Omega)$ acting as $\mathcal{H}_\alpha u = -\Delta u$ on the functions $u \in W^2_2(\Omega)$ satisfying the non-Hermitian boundary conditions

$$\Big(\frac{\partial}{\partial x_2} + \mathrm{i}\alpha\Big)u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.1}$$

It was shown in [1] that this operator is m-sectorial, densely defined, and $\mathcal{PT}$-symmetric, namely,

$$\mathcal{PT}\mathcal{H}_\alpha = \mathcal{H}_\alpha\mathcal{PT}, \tag{1.2}$$

where $(\mathcal{P}u)(x) = u(x_1, -x_2)$, and $\mathcal{T}$ is the operator of complex conjugation, $\mathcal{T}u = \overline{u}$. It was also proven in [1] that

$$\mathcal{H}^*_\alpha = \mathcal{H}_{-\alpha}, \qquad \mathcal{H}^*_\alpha = \mathcal{T}\mathcal{H}_\alpha\mathcal{T} = \mathcal{P}\mathcal{H}_\alpha\mathcal{P}. \tag{1.3}$$

A non-trivial question related to $\mathcal{H}_\alpha$ is the behavior of its eigenvalues. As $\alpha(x_1)$ is a small regular localized perturbation of a constant function, sufficient conditions were obtained in [1] for the existence and absence of isolated eigenvalues near the threshold of the essential spectrum. Similar results for both regularly and singularly perturbed models were obtained in [2–6].

Numerical experiments performed in [6, 7] provided a very non-trivial picture of the distribution of the eigenvalues. An interesting phenomenon discovered numerically in [6, 7] was the eigenvalue collision. Namely, let $t \in \mathbb{R}$ be a parameter, then as t increases, operator $\mathcal{H}_{t\alpha}$ can have two simple real isolated eigenvalues meeting at some point. Then two cases are possible. In the first of them, these eigenvalues stay real as t increases and they just pass along the real line. In the second case, the eigenvalues become complex as t increases and they are located symmetrically w.r.t. the real axis. The present paper is devoted to an analytic study of this phenomenon.

Suppose $\lambda_0 \in \mathbb{R}$ is an isolated eigenvalue of $\mathcal{H}_\alpha$, ε is a small real parameter, $\beta \in W^2_\infty(\mathbb{R})$ is some function. Denote $\Gamma_\pm := \{x : x_2 = \pm d\}$. Our first main result describes the case when λ_0 is an eigenvalue of geometric multiplicity two.

Theorem 1.1. *Assume $\lambda_0 \in \mathbb{R}$ is a double eigenvalue of $\mathcal{H}_\alpha$, $\psi_0^\pm$ are the associated eigenfunctions satisfying*

$$(\psi_0^\pm, \mathcal{T}\psi_0^\pm)_{L_2(\Omega)} = 1, \qquad (\psi_0^+, \mathcal{T}\psi_0^-)_{L_2(\Omega)} = 0. \tag{1.4}$$

Suppose also

$$(b_{11} - b_{22})^2 + 4b_{12}^2 \neq 0, \tag{1.5}$$

$$b_{11} = \mathrm{i}\int_{\Gamma_+} \beta(\psi_0^+)^2\,dx_1 - \mathrm{i}\int_{\Gamma_-} \beta(\psi_0^+)^2\,dx_1, \qquad b_{22} = \mathrm{i}\int_{\Gamma_+} \beta(\psi_0^-)^2\,dx_1 - \mathrm{i}\int_{\Gamma_-} \beta(\psi_0^-)^2\,dx_1,$$

$$b_{12} = \mathrm{i}\int_{\Gamma_+} \beta\psi_0^+\psi_0^-\,dx_1 - \mathrm{i}\int_{\Gamma_-} \beta\psi_0^+\psi_0^-\,dx_1. \tag{1.6}$$

Then for all sufficiently small ε the operator $\mathcal{H}_{\alpha+\varepsilon\beta}$ has two simple isolated eigenvalues $\lambda_\varepsilon^\pm$ converging to λ_0 as $\varepsilon \to 0$. These eigenvalues are holomorphic w.r.t. ε and the first terms of their Taylor series are

$$\lambda_\varepsilon^\pm = \lambda_0 + \varepsilon\lambda_1^\pm + \mathcal{O}(\varepsilon^2), \qquad \lambda_1^\pm = \frac{1}{2}(b_{11}+b_{22}) \pm \frac{1}{2}\big((b_{11}-b_{22})^2 + 4b_{12}^2\big)^{1/2}. \tag{1.7}$$

The second main result is devoted to the case when the geometric multiplicity of λ_0 is one but the algebraic multiplicity is two.

Theorem 1.2. *Let $\lambda_0 \in \mathbb{R}$ be a simple eigenvalue of $\mathcal{H}_\alpha$ and let ψ_0 be the associated eigenfunction. Assume that the equation*

$$(\mathcal{H}_\alpha - \lambda_0)\phi_0 = \psi_0 \tag{1.8}$$

is solvable and there exists a solution satisfying

$$(\phi_0, \mathcal{T}\psi_0)_{L_2(\Omega)} \neq 0, \quad (\phi_0, \psi_0)_{L_2(\Omega)} = 0. \tag{1.9}$$

Then eigenfunction ψ_0 can be chosen so that

$$(\phi_0, \mathcal{T}\psi_0)_{L_2(\Omega)} = 1, \qquad (\phi_0, \psi_0)_{L_2(\Omega)} = 0, \tag{1.10}$$

$$\psi_0 = \mathcal{P}\mathcal{T}\psi_0, \qquad \phi_0 = \mathcal{P}\mathcal{T}\phi_0. \tag{1.11}$$

Suppose then that this eigenfunction obeys the inequality

$$\int_{\Gamma_+} \beta \operatorname{Re}\psi_0 \operatorname{Im}\psi_0 \, dx_1 \neq 0. \tag{1.12}$$

Then for all sufficiently small ε the operator $\mathcal{H}_{\alpha+\varepsilon\beta}$ has two simple isolated eigenvalues $\lambda_\varepsilon^\pm$ converging to λ_0 as $\varepsilon \to 0$. These eigenvalues are real as

$$\varepsilon \int_{\Gamma_+} \beta \operatorname{Re}\psi_0 \operatorname{Im}\psi_0 \, dx_1 < 0 \tag{1.13}$$

and are complex as

$$\varepsilon \int_{\Gamma_+} \beta \operatorname{Re}\psi_0 \operatorname{Im}\psi_0 \, dx_1 > 0. \tag{1.14}$$

Eigenvalues $\lambda_\varepsilon^\pm$ are holomorphic w.r.t. $\varepsilon^{1/2}$ and the first terms of their Taylor series read as

$$\lambda_\varepsilon^\pm = \lambda_0 + \varepsilon^{1/2}\lambda_{1/2}^\pm + \mathcal{O}(\varepsilon), \qquad \lambda_{1/2}^\pm = \pm 2\bigg(-\int_{\Gamma_+} \beta \operatorname{Re}\psi_0 \operatorname{Im}\psi_0 \, dx_1\bigg)^{1/2}. \tag{1.15}$$

Let us discuss the results of these theorems. The typical situation of the eigenvalue collision is that two simple eigenvalues of $\mathcal{H}_{\alpha+\varepsilon\beta}$ converge to the same limiting eigenvalue λ_0 of $\mathcal{H}_\alpha$ as $\varepsilon \to 0$. Then it is a general fact from the regular perturbation theory that the algebraic multiplicity of λ_0 should be two. The above theorems address two possible situations. In the first of them the geometric multiplicity of λ_0 is two, i.e., there exist two associated linearly independent eigenfunctions. As we see from Theorem 1.1, in this situation the perturbed eigenvalues are holomorphic w.r.t. ε and their first terms in the Taylor series are given by (1.7 right). The numbers $\lambda_1^\pm$ are some fixed constants and they can be either complex or real. But an important issue is that here when changing the sign of ε, the eigenvalues can not bifurcate from real line to the complex plane or vice versa. This fact is implied by (1.7 right), namely, if $\lambda_1^\pm$ are complex numbers, then $\lambda_\varepsilon^\pm$ are also complex for both $\varepsilon < 0$ and $\varepsilon > 0$. Thus, in this case we do not face the above-mentioned phenomenon of the eigenvalue collision discovered numerically in [6], [7]. If $\lambda_1^\pm$ are real, then we need to calculate the next terms of their Taylor series to see whether they are complex or real. Once all the terms in the Taylor series are real, we deal with two real eigenvalues which just pass one through the other staying on the real line. Nevertheless, in view of formulae (1.6) we believe that choosing appropriate β we can get almost any value for the quantity in (1.5). In a particular interesting case $\beta = \alpha$ the author does not know a way of identifying the sign of $(b_{11} - b_{22})^2 + 4b_{12}^2$ or proving the reality of the eigenvalues $\lambda_\varepsilon^\pm$.

Theorem 1.2 treats the case when the geometric multiplicity of λ_0 is one. Then the Taylor series for the perturbed eigenvalues are completely different from Theorem 1.1 and here the expansions are made w.r.t. $\varepsilon^{1/2}$. And the presence of this power perfectly explains the studied phenomenon. Namely, once ε is positive, the same is true for $\varepsilon^{1/2}$, while for negative ε the square root $\varepsilon^{1/2}$ is pure imaginary. This is exactly what is needed, once ε changes the sign, real eigenvalues become complex and vice versa. Unfortunately, we cannot even analytically prove for our model the existence of such eigenvalues. We can just state that once λ_0 has geometric multiplicity one and the associated eigenfunction ψ_0 satisfies the identity $(\psi_0, \mathcal{T}\psi_0)_{L_2(\Omega)} = 0$, then equation (1.8) is solvable (see Lemma 2.1). And numerical results in [6], [7] show that this is quite a typical situation.

Our next main result provides another criterion identifying the solvability of equation (1.8)

Theorem 1.3. *Suppose λ_0 is a simple eigenvalue of $\mathcal{H}_\alpha$, the associated eigenfunction satisfies the estimate*

$$\sum_{\substack{\gamma\in\mathbb{Z}_+^2\\ |\gamma|\leqslant 2}} \Big|\frac{\partial^\gamma \psi_0}{\partial x^\gamma}(x)\Big| \leqslant \frac{C}{1+|x_1|^3}, \quad x\in\Omega. \tag{1.16}$$

Then equation (1.8) is solvable if and only if

$$\int_{\mathbb{R}^2} K(x_1,y_1)\big(\alpha(x_1)-\alpha(y_1)\big)\operatorname{Re}\psi_0(x_1,d)\operatorname{Im}\psi_0(y_1,d)\,dx_1\,dy_1=0, \tag{1.17}$$

where

$$K(x_1,y_1):=\begin{cases} x_1 & \text{if } y_1<x_1,\\ -y_1 & \text{if } y_1>x_1.\end{cases}$$

Here ψ_0 is chosen so that it satisfies the first identity in (1.11).

Assumption (1.16) is not very restrictive since usually eigenfunctions associated with isolated eigenvalues of elliptic operators decay exponentially at infinity. The main condition here is (1.17). As we shall show later in Lemma 2.1, equation (1.8) is solvable if and only if $(\psi_0,\mathcal{T}\psi_0)_{L_2(\Omega)}=0$. And we rewrite this identity to (1.17) by calculating $(\psi_0,\mathcal{T}\psi_0)_{L_2(\Omega)}$. The left hand side in (1.17) is simpler in the sense that it involves only boundary integrals while $(\psi_0,\mathcal{T}\psi_0)_{L_2(\Omega)}$ is in fact the integral over the whole strip Ω.

2. Proofs of main results

In $L_2(\Omega)$ we introduce the unitary operator $(\mathcal{U}_{\varepsilon\beta}f)(x):=\mathrm{e}^{-\mathrm{i}\varepsilon\beta(x_1)x_2}f(x)$. Then it is easy to see that the spectra of $\mathcal{H}_{\alpha+\varepsilon\beta}$ and $\mathcal{U}_{\varepsilon\beta}^{-1}\mathcal{H}_{\alpha+\varepsilon\beta}\mathcal{U}_{\varepsilon\beta}$ coincide and

$$\mathcal{U}_{\varepsilon\beta}^{-1}\mathcal{H}_{\alpha+\varepsilon\beta}\mathcal{U}_{\varepsilon\beta}=\mathcal{H}_\alpha-\varepsilon\mathcal{L}_\varepsilon, \tag{2.1}$$

$$\mathcal{L}_\varepsilon:=-2\mathrm{i}\beta' x_2\frac{\partial}{\partial x_1}-2\mathrm{i}\beta\frac{\partial}{\partial x_2}-\varepsilon\beta^2-\varepsilon(\beta')^2x_2-\mathrm{i}\beta'' x_2. \tag{2.2}$$

In the proofs of the main results we shall make use of several auxiliary lemmata.

Lemma 2.1. *Under the hypothesis of Theorem 1.2 the equation*

$$(\mathcal{H}_\alpha-\lambda_0)u=f \tag{2.3}$$

is solvable if and only if

$$(f,\mathcal{T}\psi_0)_{L_2(\Omega)}=0. \tag{2.4}$$

Under the hypothesis of Theorem 1.1 equation (2.3) is solvable if and only if

$$(f,\mathcal{T}\psi_0^\pm)_{L_2(\Omega)}=0. \tag{2.5}$$

Proof. By (1.3) we see that under the hypotheses of both Theorems 1.1 and 1.2, λ_0 is an eigenvalue of $\mathcal{H}_\alpha^*$ with the associated eigenfunction(s) $\mathcal{T}\psi_0$ or $\mathcal{T}\psi_0^\pm$. Then the lemma follows from [8, Ch. III, Sec. 6.6, Rem. 6.23]. ■

Lemma 2.2. *Suppose the hypothesis of Theorem 1.2. Then eigenfunction ψ_0 can be chosen so that relations (1.10), (1.11), and*

$$(\psi_0,\mathcal{T}\psi_0)_{L_2(\Omega)}=0 \tag{2.6}$$

hold true. The functions $\operatorname{Re}\psi_0$ and $\operatorname{Re}\phi_0$ are even w.r.t. x_2 and $\operatorname{Im}\psi_0$ and $\operatorname{Im}\phi_0$ are odd w.r.t. x_2.

Proof. Identity (2.6) follows directly from (2.4) applied to equation (1.8). Since λ_0 is a real simple eigenvalue and equation (1.8) has a unique solution satisfying the second identity in (1.10), by (1.2) we have (1.11) and thus $\operatorname{Re}\psi_0$ and $\operatorname{Re}\phi_0$ are even, while $\operatorname{Im}\psi_0$ and $\operatorname{Im}\phi_0$ are odd w.r.t. x_2. Employing this fact and (1.8), we obtain

$$\begin{aligned}(\phi_0,\mathcal{T}\psi_0)_{L_2(\Omega)}&=-\int_\Omega\phi_0(\Delta+\lambda_0)\phi_0\,dx=\mathrm{i}\int_{\Gamma_+}\alpha\phi_0^2\,dx_1-\mathrm{i}\int_{\Gamma_-}\alpha\phi_0^2\,dx_1+\int_\Omega\Big(\Big(\frac{\partial\phi_0}{\partial x_1}\Big)^2+\Big(\frac{\partial\phi_0}{\partial x_2}\Big)^2-\lambda_0\phi_0^2\Big)dx\\ &=-4\int_{\Gamma_+}\alpha\operatorname{Re}\phi_0\operatorname{Im}\phi_0\,dx_1+\int_\Omega\big(|\nabla\operatorname{Re}\phi_0|^2-|\nabla\operatorname{Im}\phi_0|^2\big)dx-\lambda_0\int_\Omega\big(|\operatorname{Re}\phi_0|^2-|\operatorname{Im}\phi_0|^2\big)dx\in\mathbb{R}.\end{aligned} \tag{2.7}$$

Hence, multiplying function ψ_0 and ϕ_0 by an appropriate constant, we can easily get the first identity in (1.10) not spoiling other established properties of ϕ_0 and ψ_0. ■

Lemma 2.3. *Suppose the hypothesis of Theorem 1.2. Then for λ close to λ_0 the resolvent $(\mathcal{H}_\alpha-\lambda)^{-1}$ can be represented as*

$$(\mathcal{H}_\alpha-\lambda)^{-1}=\frac{\mathcal{P}_{-2}}{(\lambda-\lambda_0)^2}+\frac{\mathcal{P}_{-1}}{\lambda-\lambda_0}+\mathcal{R}_\alpha(\lambda), \tag{2.8}$$

$$\mathcal{P}_{-2}=\psi_0\ell_2,\qquad \mathcal{P}_{-1}=\phi_0\ell_2+\psi_0\ell_1,\qquad \ell_2 f:=-(f,\mathcal{T}\psi_0)_{L_2(\Omega)},\qquad \ell_1 f:=-\big(f,\mathcal{T}(\phi_0-\psi_0)\big)_{L_2(\Omega)}, \tag{2.9}$$

where $\mathcal{R}_\alpha(\lambda)$ is the reduced resolvent which is a bounded and holomorphic in the λ operator.

Proof. We know by [8, Ch. III, Sec. 6.5] (see also the remark on space $\mathbf{M}'(0)$ in the proof of Theorem 1.7 in [8, Ch. VII, Sec. 1.3]) that $(\mathcal{H}_\alpha-\lambda)^{-1}$ can be expanded into the Laurent series

$$(\mathcal{H}_\alpha-\lambda)^{-1}=\sum_{n=1}^{N}\frac{\mathcal{P}_{-n}}{(\lambda-\lambda_0)^n}+\mathcal{R}_\alpha(\lambda),$$

where N is a fixed number independent of λ, $\mathcal{R}_\alpha$ is the reduced resolvent which is a bounded and holomorphic in λ operator. Given any $f\in L_2(\Omega)$, we then have

$$u=(\mathcal{H}_\alpha-\lambda)^{-1}f=\sum_{n=1}^{N}\frac{u_{-n}}{(\lambda-\lambda_0)^n}+\sum_{n=0}^{\infty}(\lambda-\lambda_0)^n u_n.$$

We substitute this formula into the equation $(\mathcal{H}_\alpha-\lambda)u=f$ and equate the coefficients at the like powers of $(\lambda-\lambda_0)$:

$$\begin{gathered}(\mathcal{H}_\alpha-\lambda_0)u_{-N}=0,\quad (\mathcal{H}_\alpha-\lambda_0)u_{-k}=u_{-k-1},\quad k=1,\dots,N-1,\\ (\mathcal{H}_\alpha-\lambda_0)u_0=f+u_{-1},\qquad (\mathcal{H}_\alpha-\lambda_0)u_1=u_0.\end{gathered} \tag{2.10}$$

This implies that $u_{-N}=\psi_0\ell_2 f$, $u_{-N+1}=\phi_0\ell_2 f+\psi_0\ell_1 f$, where ℓ_i are some functionals on $L_2(\Omega)$. If $N>2$, then by (1.9) and Lemma 2.1 the equation for u_{-N+2} is unsolvable. Hence, we can assume $N=2$. Writing then the solvability condition (2.4) for equations (2.10) and taking into consideration the identity in (1.10), we arrive easily to the formula for ℓ_2 in (2.9) and

$$\ell_1 f:=-(U_0,\mathcal{T}\psi_0)_{L_2(\Omega)}, \tag{2.11}$$

where U_0 is the solution to the equation

$$(\mathcal{H}_\alpha-\lambda_0)U_0=f+\psi_0\ell_2 f \tag{2.12}$$

satisfying

$$(U_0,\psi_0)_{L_2(\Omega)}=0. \tag{2.13}$$

It follows from (1.3) and (1.8) that

$$\begin{aligned}(U_0,\mathcal{T}\psi_0)_{L_2(\Omega)}&=\big(U_0,\mathcal{T}(\mathcal{H}_\alpha-\lambda_0)\phi_0\big)_{L_2(\Omega)}=\big(U_0,(\mathcal{H}_\alpha-\lambda_0)^*\mathcal{T}\phi_0\big)_{L_2(\Omega)}\\ &=\big((\mathcal{H}_\alpha-\lambda_0)U_0,\mathcal{T}\phi_0\big)_{L_2(\Omega)}=(f+\psi_0\ell_2 f,\mathcal{T}\phi_0)_{L_2(\Omega)}.\end{aligned}$$

These identities, the above obtained formula for ℓ_2, and (2.6), (2.11) imply formula (2.12) for ℓ_1. ∎

Lemma 2.4. *Suppose the hypothesis of Theorem 1.1. Then for λ close to λ_0 the resolvent $(\mathcal{H}_\alpha-\lambda)^{-1}$ can be represented as*

$$(\mathcal{H}_\alpha-\lambda)^{-1}=\frac{\mathcal{P}_{-1}}{\lambda-\lambda_0}+\mathcal{R}_\alpha(\lambda), \tag{2.14}$$

$$\mathcal{P}_{-1}=\psi_0^+\ell_+ +\psi_0^-\ell_-,\qquad \ell_\pm f:=-(f,\mathcal{T}\psi_0^\pm)_{L_2(\Omega)}, \tag{2.15}$$

where $\mathcal{R}_\alpha(\lambda)$ is the reduced resolvent which is a bounded and holomorphic in λ operator.

The proof of this lemma is similar to that of Lemma 2.3, we just should bear in mind that due to (1.4) and Lemma 2.1 the equations

$$(\mathcal{H}_\alpha-\lambda_0)u=\psi_0^\pm$$

are unsolvable.

We proceed to the proofs of Theorems 1.1, 1.2, 1.3.

Proof of Theorem 1.2. The proof is based on the modified version of the Birman-Schwinger principle suggested in [9] in the form developed in [10]. In view of (2.1), the eigenvalue equation for $\mathcal{H}_{\alpha+\varepsilon\beta}$ is equivalent to the same equation for $\mathcal{H}_\alpha - \varepsilon\mathcal{L}_\varepsilon$. The latter equation can be written as

$$(\mathcal{H}_\alpha - \lambda_\varepsilon)\psi_\varepsilon = \varepsilon\mathcal{L}_\varepsilon\psi_\varepsilon. \tag{2.16}$$

We then invert the operator $(\mathcal{H}_\alpha - \lambda_\varepsilon)$ by Lemma 2.3 and obtain

$$\psi_\varepsilon = \varepsilon\frac{\mathcal{P}_{-2}\mathcal{L}_\varepsilon\psi_\varepsilon}{(\lambda_\varepsilon-\lambda_0)^2} + \varepsilon\frac{\mathcal{P}_{-1}\mathcal{L}_\varepsilon\psi_\varepsilon}{\lambda_\varepsilon-\lambda_0} + \varepsilon\mathcal{R}_\alpha(\lambda_\varepsilon)\psi_\varepsilon.$$

By Lemma 2.3 the operator $\mathcal{R}_\alpha(\lambda)$ is bounded uniformly in λ close to λ_0 and hence the inverse $\mathcal{A}(z,\varepsilon) := \big(\mathrm{I} - \varepsilon\mathcal{R}_\alpha(\lambda_0+z)\big)^{-1}$ is well-defined and is uniformly bounded for all λ close to λ_0 and for all sufficiently small ε. We apply this operator to the latter equation and get

$$\psi_\varepsilon = \frac{\varepsilon}{z_\varepsilon^2}\mathcal{A}(\lambda_0+z_\varepsilon,\varepsilon)\mathcal{P}_{-2}\mathcal{L}_\varepsilon\psi_\varepsilon + \frac{\varepsilon}{z_\varepsilon}\mathcal{A}(\lambda_0+z_\varepsilon,\varepsilon)\mathcal{P}_{-1}\mathcal{L}_\varepsilon\psi_\varepsilon, \tag{2.17}$$

where we denote $z_\varepsilon := \lambda_\varepsilon - \lambda_0$. Then we apply functionals $\ell_2\mathcal{L}_\varepsilon$, $\ell_1\mathcal{L}_\varepsilon$ to the obtained equation and it results in

$$\begin{aligned}
&\Big(\frac{\varepsilon}{z_\varepsilon}A_{11}(z_\varepsilon,\varepsilon) - 1\Big)X_1 + \frac{\varepsilon}{z_\varepsilon^2}\big(A_{11}(z_\varepsilon,\varepsilon) + z_\varepsilon A_{12}(z_\varepsilon,\varepsilon)\big)X_2 = 0,\\
&\frac{\varepsilon}{z_\varepsilon}A_{21}(z_\varepsilon,\varepsilon)X_1 + \Big(\frac{\varepsilon}{z_\varepsilon^2}\big(A_{21}(z_\varepsilon,\varepsilon) + z_\varepsilon A_{22}(z_\varepsilon,\varepsilon)\big) - 1\Big)X_2 = 0,
\end{aligned} \tag{2.18}$$

where $X_i = \ell_i\mathcal{L}_\varepsilon\psi_\varepsilon$, and

$$A_{i1}(z,\varepsilon) := \ell_i\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\psi_0, \quad A_{i2}(z,\varepsilon) := \ell_i\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\phi_0, \quad i=1,2.$$

The obtained system of equations is linear w.r.t. (X_1,X_2). We need a non-zero solution to this system since otherwise by (2.17) we would get $\psi_\varepsilon = 0$ and ψ_ε then cannot be an eigenfunction. System (2.18) has a nonzero solution if its determinant vanishes. It implies the equation

$$z_\varepsilon^2 - \varepsilon\big(A_{11}(z_\varepsilon,\varepsilon) + A_{22}(z_\varepsilon,\varepsilon)\big)z_\varepsilon - \varepsilon A_{21}(z_\varepsilon,\varepsilon) + \varepsilon^2\big(A_{11}(z_\varepsilon,\varepsilon)A_{22}(z_\varepsilon,\varepsilon) - A_{12}(z_\varepsilon,\varepsilon)A_{21}(z_\varepsilon,\varepsilon)\big) = 0,$$

which is equivalent to the following two

$$z_\varepsilon = G_\pm(z_\varepsilon,\varepsilon^{1/2}), \tag{2.19}$$

where

$$\begin{aligned}
G_\pm(z,\varkappa) := {}& \frac{\varkappa^2(A_{11}(z,\varkappa^2) + A_{22}(z,\varkappa^2))}{2}\\
&\pm \varkappa\Big(A_{21}(z,\varkappa^2) + \frac{\varkappa^2}{4}\big(A_{11}(z,\varkappa^2) - A_{22}(z,\varkappa^2)\big)^2 + \varkappa^2 A_{12}(z,\varkappa^2)A_{21}(z,\varkappa^2)\Big)^{1/2}.
\end{aligned} \tag{2.20}$$

Here the branch of the square root is fixed by the restriction $1^{1/2} = 1$. It is clear that the functions A_{ij} are jointly holomorphic w.r.t. sufficiently small z and ε. Moreover, by (2.2)

$$A_{21}(0,\varepsilon) = \ell_2\mathcal{L}_\varepsilon\mathcal{A}(0,\varepsilon)\psi_0 = \mathrm{i}\ell_2\Big(-2\beta' x_2\frac{\partial}{\partial x_1} - 2\beta\frac{\partial}{\partial x_2} - \beta'' x_2\Big)\psi_0 + \mathcal{O}(\varepsilon). \tag{2.21}$$

To calculate the first term on the right hand side of this identity, we first observe that by the equation for ψ_0 we have

$$-\Big(2\beta' x_2\frac{\partial}{\partial x_1} + 2\beta\frac{\partial}{\partial x_2} + \beta'' x_2\Big)\psi_0 = -(\Delta+\lambda_0)\beta x_2\psi_0 =: g.$$

Now we find $\mathrm{i}\ell_2 g$ by integration by parts

$$\begin{aligned}
\mathrm{i}\ell_2 g = \int_\Omega \psi_0(\Delta+\lambda_0)\beta x_2\psi_0\,dx = {}& \mathrm{i}\int_{\Gamma_+}\Big(\psi_0\frac{\partial}{\partial x_2}\beta x_2\psi_0 - \beta x_2\psi_0\frac{\partial\psi_0}{\partial x_2}\Big)\,dx_1\\
&- \mathrm{i}\int_{\Gamma_-}\Big(\psi_0\frac{\partial}{\partial x_2}\beta x_2\psi_0 - \beta x_2\psi_0\frac{\partial\psi_0}{\partial x_2}\Big)\,dx_1 = \mathrm{i}\int_{\Gamma_+}\beta\psi_0^2\,dx_1 - \mathrm{i}\int_{\Gamma_-}\beta\psi_0^2\,dx_1.
\end{aligned} \tag{2.22}$$

Together with Lemma 2.2 this implies

$$\mathrm{i}\ell_2 g = -4\int_{\Gamma_+}\beta\,\mathrm{Re}\,\psi_0\,\mathrm{Im}\,\psi_0\,dx_1. \tag{2.23}$$

Hence, by (2.20), (2.22), (1.12), and the properties of functions A_{ij} we conclude that functions $G_\pm$ are jointly holomorphic w.r.t. sufficiently small z and $\varkappa$. Applying the Rouché theorem as in [10, Sec. 4], we conclude that for all sufficiently small $\varkappa$ each of the functions $z \mapsto z - G_\pm(z,\varkappa)$ has a simple zero $z_\pm(\varkappa)$ in a small neighborhood of the origin. By the implicit function theorem these zeroes are holomorphic w.r.t. $\varkappa$. Thus, the desired solutions to equations (2.19) are $z_\pm(\varepsilon^{1/2})$, and these functions are holomorphic w.r.t. $\varepsilon^{1/2}$. Moreover, it follows from (2.19), (2.20), (2.21), (2.22), (2.23) that

$$z_\pm(\varepsilon^{1/2}) = G_\pm(0,\varepsilon^{1/2}) + \mathcal{O}(\varepsilon) = \pm\varepsilon^{1/2} A_{21}^{1/2}(0,\varepsilon) + \mathcal{O}(\varepsilon)$$

and then the sought eigenvalues are $\lambda_\varepsilon^\pm = \lambda_0 + z_\pm(\varepsilon^{1/2})$. These eigenvalues are holomorphic w.r.t. $\varepsilon^{1/2}$ and obey (1.15). Let us prove that these eigenvalues are real as (1.13) holds true and are complex once (1.14) is satisfied. The latter statement follows easily from formulae (1.15) since in this case $\varepsilon^{1/2}\lambda_{1/2}^\pm$ are two imaginary numbers. To prove the reality, as one can easily make sure, it is sufficient to prove that functions $G_\pm(z,\varkappa)$ are real for real z and $\varkappa$. Then the existence of a real root is implied easily by the implicit function theorem for real functions.

In view of definition (2.20) of $G_\pm$, the desired fact is yielded by the similar reality of A_{ij}. Let us prove the latter.

It follows from Lemma 2.3 that for each $f \in L_2(\Omega)$ the function

$$\mathcal{R}_\alpha(\lambda) f = (\mathcal{H}_\alpha - \lambda)^{-1} f - \frac{\mathcal{P}_{-2} f}{(\lambda-\lambda_0)^2} - \frac{\mathcal{P}_{-1} f}{\lambda - \lambda_0}$$

solves the equation

$$(\mathcal{H}_\alpha - \lambda)\mathcal{R}_\alpha(\lambda) f = f + \psi_0 \ell_1 f + \phi_0 \ell_2 f. \tag{2.24}$$

Employing definition (2.2) of $\mathcal{L}_\varepsilon$, we check easily that $\mathcal{PTL}_\varepsilon = \mathcal{L}_\varepsilon\mathcal{PT}$. This identity and (1.11), (2.24) yield that for $z\in\mathbb{R}$, $\varkappa\in\mathbb{R}$

$$\mathcal{PTL}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\psi_0 = \mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\psi_0, \qquad \mathcal{PTL}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\phi_0 = \mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\phi_0.$$

Using (1.11) once again, for $z\in\mathbb{R}$, $\varkappa\in\mathbb{R}$ we get

$$\overline{A_{11}(z,\varkappa)} = \big(\mathcal{PTL}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\psi_0, \mathcal{P}\psi_0\big)_{L_2(\Omega)} = \big(\mathcal{TL}_\varepsilon\mathcal{A}(\lambda_0+z,\varkappa)\psi_0, \mathcal{T}\psi_0\big)_{L_2(\Omega)} = A_{11}(z,\varkappa).$$

The reality of other functions A_{ij} can be proven in the same way. The proof is complete. ∎

Proof of Theorem 1.1. The main ideas here are the same as in the proof of Theorem 1.2, so, we focus only on the main milestones. We again begin with (2.1) and invert $(\mathcal{H}_\varepsilon - \lambda_\varepsilon)$ by Lemma 2.2. It leads us to an analogue of equation (2.17),

$$\psi_\varepsilon = \frac{\varepsilon}{z_\varepsilon}\mathcal{A}(\lambda_0 + z_\varepsilon, \varepsilon)\mathcal{P}_{-1}\mathcal{L}_\varepsilon\psi_\varepsilon, \tag{2.25}$$

where operator $\mathcal{A}$ is introduced in the same way as above. We then apply functionals $\ell_\pm\mathcal{L}_\varepsilon$ to this equation

$$\Big(\frac{\varepsilon}{z_\varepsilon}B_{11}(z_\varepsilon,\varepsilon) - 1\Big)X_1 + \frac{\varepsilon}{z_\varepsilon}B_{12}(z_\varepsilon,\varepsilon)X_2 = 0, \qquad \frac{\varepsilon}{z_\varepsilon}B_{21}(z_\varepsilon,\varepsilon)X_1 + \Big(\frac{\varepsilon}{z_\varepsilon}B_{22}(z_\varepsilon,\varepsilon) - 1\Big)X_2 = 0, \tag{2.26}$$

$$B_{11}(z,\varepsilon) := \ell_+\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\psi_0^+, \qquad B_{12}(z,\varepsilon) := \ell_+\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\psi_0^-,$$
$$B_{21}(z,\varepsilon) := \ell_-\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\psi_0^+, \qquad B_{22}(z,\varepsilon) := \ell_-\mathcal{L}_\varepsilon\mathcal{A}(\lambda_0+z,\varepsilon)\psi_0^-.$$

The determinant of system (2.26) should again vanish and it implies the equation

$$z_\varepsilon^2 - \varepsilon\big(B_{11}(z_\varepsilon,\varepsilon) + B_{22}(z_\varepsilon,\varepsilon)\big) + \varepsilon^2\big(B_{11}(z_\varepsilon,\varepsilon)B_{22}(z_\varepsilon,\varepsilon) - B_{12}(z_\varepsilon,\varepsilon)B_{21}(z_\varepsilon,\varepsilon)\big) = 0,$$

which splits into other two

$$z_\varepsilon = Q_\pm(z_\varepsilon,\varepsilon), \tag{2.27}$$
$$Q_\pm(z,\varepsilon) := \frac{\varepsilon}{2}\big(B_{11}(z_\varepsilon,\varepsilon) + B_{22}(z_\varepsilon,\varepsilon)\big) \pm \frac{\varepsilon}{2}\big((B_{11}(z,\varepsilon) - B_{22}(z,\varepsilon))^2 + 4B_{12}(z,\varepsilon)B_{21}(z,\varepsilon)\big)^{1/2}.$$

Here the branch of the square root is fixed by the restriction $1^{1/2} = 1$. Let us prove that this square root is jointly holomorphic w.r.t. z and ε. Integrating by parts as in (2.22) and employing (1.1), one can make easily sure that

$$B_{ii} = b_{ii} + \mathcal{O}(\varepsilon), \quad i=1,2, \qquad B_{12}(0,\varepsilon) = b_{12} + \mathcal{O}(\varepsilon), \qquad B_{21}(0,\varepsilon) = b_{21} + \mathcal{O}(\varepsilon). \tag{2.28}$$

Hence, by assumption (1.5), functions $Q_\pm$ are jointly holomorphic w.r.t. z and ε. Proceeding now as in the proof of Theorem 1.2, we arrive at the statement of Theorem 1.1. ∎

Proof of Theorem 1.3. Denote

$$\psi(x) := \frac{1}{2}\int_{-\infty}^{x_1} t\psi_0(t,x_2)\,dt.$$

In view of (1.16) this function is well-defined. Throughout the proof we shall deal with several integrals of such kind and all of them will be well-defined due to (1.16). In what follows we shall not stress this fact anymore.

Employing the equation for ψ_0, integrating by parts, and bearing in mind estimates (1.16), we get

$$(\Delta+\lambda_0)\psi = \psi_0 + \frac{1}{2}x_1\frac{\partial\psi_0}{\partial x_1} + \frac{1}{2}\int_{-\infty}^{x_1} t\Big(\frac{\partial^2}{\partial x_2^2}+\lambda_0\Big)\psi_0(t,x_2)\,dt = \psi_0 + \frac{1}{2}x_1\frac{\partial\psi_0}{\partial x_1} - \frac{1}{2}x_1\int_{-\infty}^{x_1}\frac{\partial^2\psi_0}{\partial x_1^2}(t,x_2)\,dt = \psi_0.$$

The proven equation for ψ allows us to integrate once again,

$$\begin{aligned}\int_\Omega \psi_0^2\,dx = \int_\Omega \psi_0(\Delta+\lambda_0)\psi\,dx = \int_{\Gamma_+}\Big(\psi_0\frac{\partial\psi}{\partial x_2} - \psi\frac{\partial\psi_0}{\partial x_2}\Big)\,dx_1 - \int_{\Gamma_-}\Big(\psi_0\frac{\partial\psi}{\partial x_2} - \psi\frac{\partial\psi_0}{\partial x_2}\Big)\,dx_1 \\ = \int_{\Gamma_+}\psi_0\Big(\frac{\partial\psi}{\partial x_2}+\mathrm{i}\alpha\psi\Big)\,dx_1 - \int_{\Gamma_-}\psi_0\Big(\frac{\partial\psi}{\partial x_2}+\mathrm{i}\alpha\psi\Big)\,dx_1.\end{aligned}$$

Now we employ identity (1.11) and boundary condition (1.1) for ψ_0 to simplify the sum of these integrals,

$$\begin{aligned}\int_\Omega \psi_0^2\,dx = &-\int_{\Gamma_+} dx_1\,\mathrm{Re}\,\psi_0(x_1,d)x_1\int_{-\infty}^{x_1}\big(\alpha(x_1)-\alpha(y_1)\big)\,\mathrm{Im}\,\psi_0(y_1,d)\,dy_1 \\ &-\int_{\Gamma_+} dx_1\,\mathrm{Im}\,\psi_0(x_1,d)x_1\int_{-\infty}^{x_1}\big(\alpha(x_1)-\alpha(y_1)\big)\,\mathrm{Re}\,\psi_0(y_1,d)\,dy_1 \\ = &-\int_{\Gamma_+} dx_1\,\mathrm{Re}\,\psi_0(x_1,d)x_1\int_{-\infty}^{x_1}\big(\alpha(x_1)-\alpha(y_1)\big)\,\mathrm{Im}\,\psi_0(y_1,d)\,dy_1 \\ &+\int_{\Gamma_+} dx_1\,\mathrm{Re}\,\psi_0(x_1,d)x_1\int_{x_1}^{+\infty}\big(\alpha(x_1)-\alpha(y_1)\big)\,\mathrm{Im}\,\psi_0(y_1,d)\,dy_1 \\ = &-\int_{\mathbb{R}^2} K(x_1,y_1)\big(\alpha(x_1)-\alpha(y_1)\big)\,\mathrm{Re}\,\psi_0(y_1,d)\,\mathrm{Im}\,\psi_0(y_1,d)\,dx_1\,dy_1.\end{aligned}$$

By (2.4) we then conclude that equation (1.8) is solvable if and only if identity (1.17) holds true. ∎

Remark 2.5. The idea of the latter proof was borrowed from the proof of Lemma 2.2 in [11], see also proof of Lemma 3.6 in [10].

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