

NEW SPECTRAL STATISTICS FOR ENSEMBLES OF 2×2 REAL SYMMETRIC RANDOM MATRICES

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ABSTRACT. We investigate spacing statistics for ensembles of various real random matrices where the matrix-elements have various Probability Distribution Functions (PDFs: $f(x)$) including Gaussian. For two modifications of 2×2 matrices with various PDFs, we derive the spacing distributions $p(s)$ of adjacent energy eigenvalues. Nevertheless, they show the linear level repulsion near $s = 0$ as αs where α depends on the choice of the PDF. More interestingly when $f(x) = xe^{-x^2}$ ($f(0) = 0$), we get cubic level repulsion near $s = 0$: $p(s) \sim s^3 e^{-s^2}$. We also derive the distribution of eigenvalues $D(\epsilon)$ for these matrices.

KEYWORDS: real symmetric matrices; Wigner surmise.

1. INTRODUCTION

Due to matrix mechanics of Heisenberg and Method of Linear Combination of Atomic Orbitals (LCAO) one can easily visualize the eigenspectrum of various systems with time-reversal symmetry as result of diagonalization of a real symmetric matrix where the matrix element are calculated using the inter-particle interaction. Most of the times this interaction is not known. For example, energy levels of various nuclei are known experimentally but the nuclear interaction is not really known. Random Matrix Theory [1–5] originated by considering the level spacing statistics $P(S)$ between eigenvalues of real symmetric matrices

$$R_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad R_2 = \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \quad (1)$$

when matrix elements a, b, c are random numbers with the Probability Distribution Function (PDF) as Gaussian: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$. The spacing of eigenvalues are given as $\mathcal{S}_1 \sim \sqrt{4b^2 + (a-c)^2}$ and $\mathcal{S}_2 \sim \sqrt{b^2 + c^2}$, respectively. Notice that the \mathcal{S}_1 , is function of three parameters (a, b, c), whereas \mathcal{S}_2 function of just two (b, c). Notwithstanding this disparity and the complexity of the multiple integral

$$P(S) = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, b, c) \delta[S - \mathcal{S}(a, b, c)] da db dc \quad (2)$$

the spacing distributions in the two cases (R_1, R_2) turned out to be the same: $P(S) = S e^{-S^2}$. When arranged to yield the average spacing as 1, the normalized spacing distributions is written as

$$p_W(s) = \frac{\pi s}{2} e^{-\pi s^2/4}, \quad (3)$$

This is called the spacing distribution of Gaussian Orthogonal Ensemble (GOE) due to the orthogonal symmetry of real symmetric matrices. Moreover p_W is well known as Wigner distribution function. Wigner surmised [1–5] that the spacing distribution of adjacent eigenvalues of N number of $n \times n$ Gaussian random real symmetric matrices will again be given by (3). Next, Wigner predicted that (3) would eventually represent the spacing statistics of neutron-nucleus scattering resonances and nuclear levels. Notice that near zero $p_W(s)$ is linear as $\pi s/4$, this is called the linear level repulsion of adjacent eigenvalues. The rotational invariance and invariance under time-reversal of a symmetric matrix lie behind the linear level repulsion. Consequently, the spacing of nuclear levels of same angular momentum J and parity π indeed display [1–5] $p(s)$ in (3). Wigner's surmise is strange but true, each nucleus behaves like a matrix of large order.

Even much later, in the recent years investigations on spacing distributions of ensembles random matrices using 2×2 continue to be an attractive proposition for both symmetric/Hermitian [7, 8] and non-Hermitian matrices [9–14]. In these works [7, 8] one has taken Gaussian distribution with zero mean and different variances for various entries of the matrices and derived a variety of spacing distributions. Similarly, under Gaussian

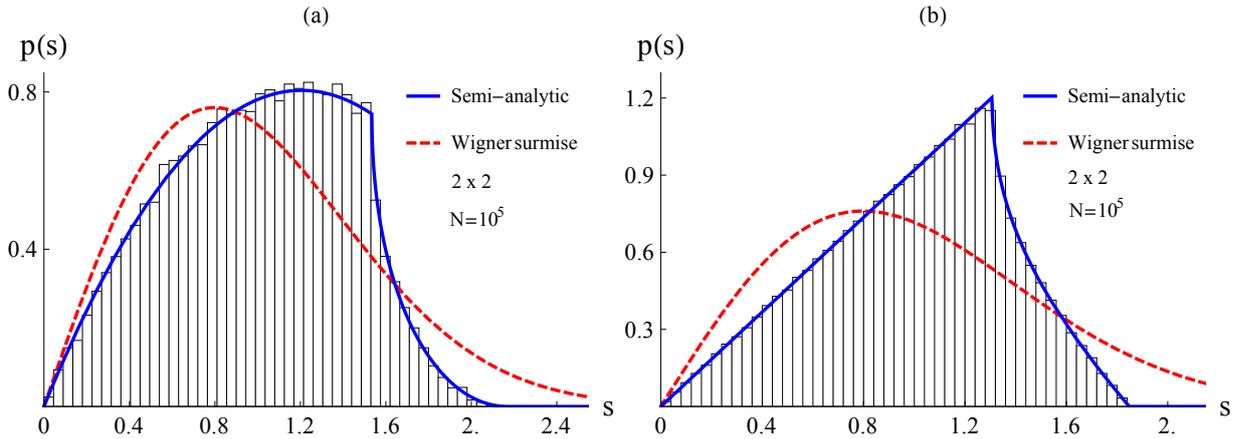


FIGURE 1. $p(s)$ for (a): R_1 and (b): R_2 matrices due to the uniform distribution of elements. The solid lines are due to (7) and (10), dashed lines represent the Wigner distribution (3). These two $p(s)$ are distinctly different but near $s = 0$ they are linear (αs), with α as 1.23 and 1.01, respectively

PDF, for ensembles of several 2×2 pseudo-symmetric and pseudo-Hermitian representing Parity-Time-reversal symmetric systems the novel expressions of $p(s)$ have been derived [9–14].

Here, we show that even two modifications of real symmetric 2×2 random matrices yield different $p(s)$ under the same probability distribution function (PDF) $f(x)$. Similarly, one type of matrix under several non-Gaussian PDFs yield distinct expressions for $p(s)$. However, all the spacing distributions display the linear (αs) level repulsion near $s = 0$ wherein α depends on the type of PDF and the type of matrix (1) being used. The question of using a non-Gaussian probability distribution to test Wigner’s second conjecture does not appear to have attracted much attention after [6]. However, it is generally believed that spectral distributions are insensitive to PDFs. Use of many non-Gaussian distributions for finding the probability [15] of occurrence of real eigenvalues for the product of n number of 2×2 real random matrices is worth mentioning.

The other interesting spectral statistics denoted as $D(\epsilon)$ is called distribution of eigenvalues of $n \times n$ real symmetric matrices when n is large. Wigner proposed it to be [1–5]

$$D(\epsilon) = \frac{2}{\pi} \sqrt{1 - \epsilon^2}, \quad \epsilon = E/E_* \tag{4}$$

which is well known as Wigner’s semicircle law. In case of large values of n , E_* is the maximum eigenvalue of the matrix. Due to the historic connection of 2×2 matrices in RMT and specially due to the astonishing sameness of $p(s)$ in case of GOE for $n = 2$ and $n \gg 2$, the question arising here is as to what is the analytic form of $D(\epsilon)$ in case of $n = 2$. Due to the numerical calculations of Porter we know that qualitatively $D(\epsilon)$ makes an interesting transition from a bell shape to the semi-circle as the n increases. In case of $n = 2$, we collect a large number (N) of matrices and find the mean of positive eigenvalues to fix $E_* = \bar{E}$ to obtain $D(\epsilon)$ both analytically and by finding their histograms numerically. The obtained $D(\epsilon)$ once again are unlike the Wigner’s semi-circle law (4) (for $n = 2$) and our analytic/semi-analytic results agree excellently with the numerically computed histograms.

In §2, we wish to present analytic or semi-analytic $p(s)$ for four non-Gaussian PDFs of elements of matrices for two types of 2×2 . These non-Gaussian PDFs are : Uniform (U), Exponential (E: $f(x) = e^{-|x|}$), Super-Gaussian (SG: $f(x) = e^{-x^4}$) and Maxwellian (M: $f(x) = xe^{-x^2}$). In §3, we derive $D(\epsilon)$ for R_1 and R_2 and plot them in Figure 4 with their numerically computed histograms.

2. ENSEMBLES OF 2×2 REAL-SYMMETRIC RANDOM MATRICES AND THEIR SPACING DISTRIBUTIONS

In this section we find $P(S)$ (2) for two real symmetric matrices R_1 and R_2 for four PDFs (U, E, SG, M). By finding the average spacing as $\bar{S} = \int_0^\infty SP(S) dS / \int_0^\infty P(S) S$, we then find the normalized spacing distribution $p(s = S/\bar{S})$. We thus conform to the invariance of distributions in two S and s as $p(s)ds = P(S)dS$.

2.1. UNIFORM DISTRIBUTION: $f(x) = 1, 0 \leq |x| \leq \lambda, f(x) = 0, |x| > \lambda$

For the matrix R_1 , we have to evaluate

$$P(S) = A \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda \int_{-\lambda}^\lambda \delta[S - \sqrt{4b^2 + (a - c)^2}] da db dc, \tag{5}$$

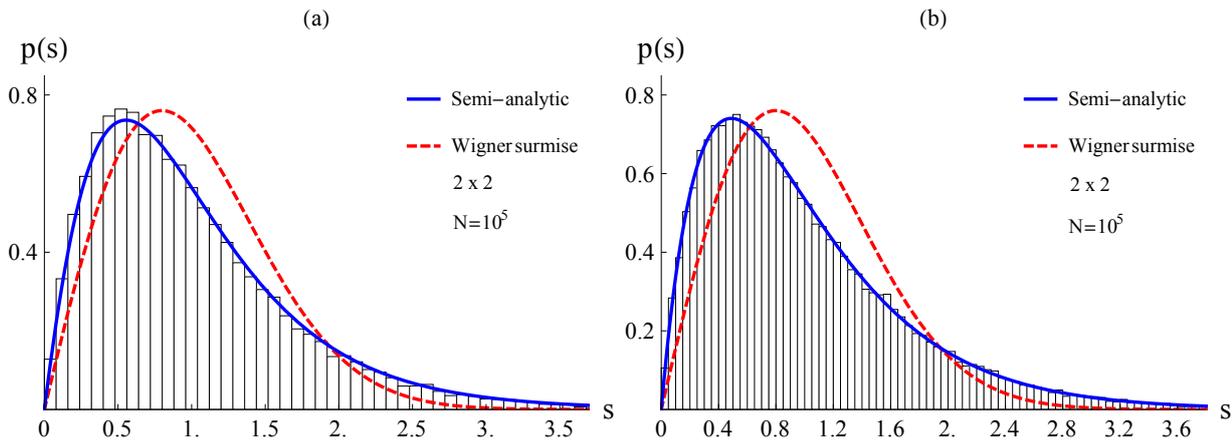


FIGURE 2. The same as in Figure 1, for Exponential PDF ($E: e^{-|x|}$) arising from semi-analytic expression (13) and semi-analytic form (15). Here α values are 4.05 and 2.91, respectively.

Without a loss of generality we may choose $\lambda = 1$. Let us introduce the transformation from (a, b, c) to (u, v, w) as $u = a - c, v = a + c, w = 2b$, (5) becomes

$$P(S) = \begin{cases} A \int_0^S du \int_0^2 dw \int_{u-2}^{2-u} \delta(S - \sqrt{w^2 + u^2}) dv, & 0 \leq S \leq 2 \\ A \int_{\sqrt{S^2-4}}^2 du \int_0^2 dw \int_{u-2}^{2-u} \delta(S - \sqrt{w^2 + u^2}) dv, & 2 < S < 2\sqrt{2} \\ 0, & S \geq 2\sqrt{2}. \end{cases} \quad (6)$$

We find that integrals in (6) can be done and $P(S)$ turns out to be a piecewise continuous function given as

$$P(S) = \begin{cases} A'S(\pi - S)/4, & 0 \leq S \leq 2, \\ A'S/2 [\sin^{-1}(2/S) - \sin^{-1}(\sqrt{S^2 - 4}/S)] + \frac{S}{4} [\sqrt{S^2 - 4} - 2], & 2 < S < 2\sqrt{2}, \\ 0, & S \geq 2\sqrt{2}. \end{cases} \quad (7)$$

For the matrix R_2 (1),

$$P(S) = A \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \int_{-\lambda}^{\lambda} \delta[S - \sqrt{b^2 + c^2}] da db dc. \quad (8)$$

The a -integral is separable and it will yield a multiplicative constant. we convert the double integral in b and c in to polar form as $b = r \cos \theta, c = r \sin \theta$

$$P(S) = \begin{cases} A' \int_0^1 \int_0^{\pi/4} r dr \delta(S - r) d\theta, & 0 \leq S \leq 1 \\ A' \int_1^{\sqrt{2}} \int_{\cos^{-1}(1/r)}^{\pi/4} r dr \delta(S - r) d\theta, & 1 < S < \sqrt{2}, \\ 0, & S \geq \sqrt{2}. \end{cases} \quad (9)$$

Finally for real symmetric matrix R_2 (1) when the elements are distributed uniformly over $[-1,1]$, from (9) we get the continuous three piece spacing distribution function for $s \in (0, \infty)$ ($\lambda = 1$) as

$$P(S) = \begin{cases} A' \pi S/2, & 0 \leq S \leq 1, \\ 2A'S[\pi/4 - \cos^{-1}(1/S)], & 1 < S < \sqrt{2}, \\ 0, & S \geq \sqrt{2}. \end{cases} \quad (10)$$

In Figure 1, we plot $p(s)$ arising from analytic results (7,10) along with the histograms generated from 100000 ($= N$), 2×2 real symmetric matrices of the types (a): R_1 and (b): R_2 . Near $s = 0$, they show linear repulsion, where α (the coefficient of linearity) is 1.23 and 1.09, respectively. Notice the excellent agreement of solid lines with histograms, the dashed lines represent Wigner's distribution (3).

2.2. EXPONENTIAL DISTRIBUTION: $f(x) = e^{-|x|}$

Multiple integrals in $P(S)$ for R_1 under exponential PDF can be written as

$$P(S) = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(|a|+|b|+|c|)} \delta[S - \sqrt{4b^2 + (a-c)^2}] da db dc. \quad (11)$$

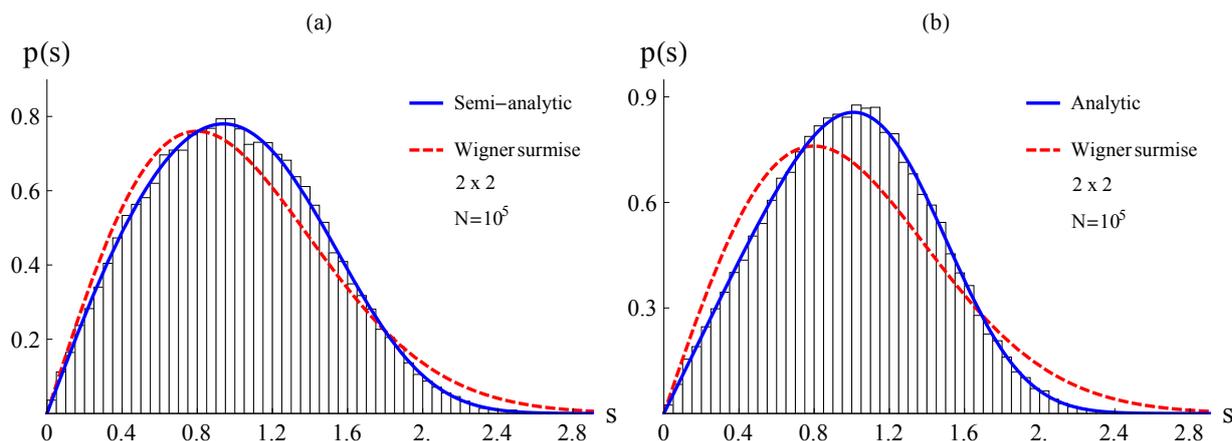


FIGURE 3. The same as in Figures 1 and 2, for super-Gaussian PDF (SG: e^{-x^4}) arising from the semi-analytic expression (18) and the analytic one (19). Here α values are 1.30 and 0.91, respectively.

We transform $P(S)$ into the three dimensional spherical polar co-ordinates using $2b = r \cos \theta$, $a = r \sin \theta \cos \phi$, $c = r \sin \theta \sin \phi$ as

$$P(S) = A \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-r(|\cos \theta|/2 + \sin \theta(|\cos \phi| + |\sin \phi|))} \delta[S - rg(\theta, \phi)] r^2 dr \sin \theta d\theta d\phi. \tag{12}$$

Crashing the delta function in above, we get a θ, ϕ integral

$$P(S) = A' \int_0^{\pi/2} \int_0^\pi e^{-S(|\cos \theta|/2 + \sin \theta(|\cos \phi| + |\sin \phi|))/g(\theta, \phi)} \frac{S^2}{|g(\theta, \phi)|^3} \sin \theta d\theta d\phi, \tag{13}$$

$$g(\theta, \phi) = \sqrt{1 - \sin^2 \theta \sin 2\phi},$$

Due to the symmetry of integrand the domains of integrations in (13) have been reduced.

The $P(S)$ of R_2 for exponential distribution can be written as

$$P(S) = A \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(|a|+|b|+|c|)} \delta[S - \sqrt{b^2 + c^2}] da db dc. \tag{14}$$

Here the a -integral is separable and gives 1. The remaining double integral can be converted to polar form as

$$P(S) = A' S \int_0^{\pi/2} e^{-S(\sin \theta + \cos \theta)} d\theta. \tag{15}$$

The integrals (13), (15) are further inexpressible in terms of known functions. $p(s)$ for these two cases are plotted in Figure 2, they look similar though distinct, notice their linear behaviour near $s = 0$ like Wigner's distribution (dashed line).

2.3. SUPER-GAUSSIAN DISTRIBUTION: $f(x) = e^{-x^4}$

For R_1 (1), the $P(S)$ integral (2) becomes

$$P(S) = A \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(a^4+b^4+c^4)} \delta[S - \sqrt{4b^2 + (a-c)^2}] da db dc. \tag{16}$$

We transform $P(S)$ into the three dimensional spherical polar co-ordinates using $2b = r \cos \theta$, $a = r \sin \theta \cos \phi$, $c = r \sin \theta \sin \phi$ as

$$P(S) = A \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-r^4(\cos^4 \theta/16 + \sin^4 \theta(\cos^4 \phi + \sin^4 \phi))} \delta[S - rg(\theta, \phi)] r^2 dr \sin \theta d\theta d\phi. \tag{17}$$

Crashing the delta function in above, we get a θ, ϕ integral

$$P(S) = A' \int_0^{\pi/2} \int_0^\pi e^{-S^4(\cos^4 \theta/16 + \sin^4 \theta(\cos^4 \phi + \sin^4 \phi))/g^4(\theta, \phi)} \frac{S^2}{|g(\theta, \phi)|^3} \sin \theta d\theta d\phi, \tag{18}$$

$$g(\theta, \phi) = \sqrt{1 - \sin^2 \theta \sin 2\phi}.$$

For the matrix R_2 , the a -integral in $P(S)$ is separable gives a multiplying constant. Then the double integral in b, c is changed to polar form where by crashing the delta function we get an integral

$$P(S) = A'S \int_0^{\pi/2} e^{-S(\cos^4 \theta + \sin^4 \theta)} d\theta = A'Se^{-3S^4/4} \int_0^{2\pi} e^{-(S^4 \cos t)/4} dt = \frac{A'\pi S}{2} e^{-3S^4/4} I_0(S^4/4), \tag{19}$$

$p(s)$ corresponding to (18) and (19) are plotted in Figure 3 showing linear level repulsion near $s = 0$.

2.4. MAXWELLIAN DISTRIBUTION: $f(x) = xe^{-x^2}$, $x > 0$

For R_1 type of real symmetric matrix $P(S)$ is not simple, however here we would like to show that when the PDF does not peak at $x = 0$, we get highly non-linear behaviour of $P(S)$ near $S = 0$ For R_2 , to this end we convert the integral (2) to polar form and get

$$P(S) = A'S^3 e^{-S^2}, \tag{20}$$

displaying nonlinear cubic behaviour $\sim S^3$ behaviour near $S = 0$. In RMT, the PDF of matrix elements is usually taken as symmetric and peaking at $x = 0$ and one gets linear level repulsion near $s = 0$. But when we take the non-symmetric Maxwellian distribution ($f(0) = 0$), we get cubic level repulsion near $s = 0$. Therefore, it would be interesting to see whether for $n \times n$ (n large) the cubic level repulsion persists.

3. DISTRIBUTION OF EIGENVALUES $D(\epsilon)$ OF 2×2 GAUSSIAN RANDOM MATRICES

We collect $2N$ eigenvalues of N 2×2 matrices to find the mean of positive eigenvalue (\bar{E}) and divide all eigenvalues by \bar{E} and find histograms $D(\epsilon)$. For a large real symmetric matrix this distribution is well known as semi-circle law (4).

The distribution of eigenvalues $E_1(a, b, c)$ and $E_2(a, b, c)$ can be obtained analytically as

$$g(E) = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a, b, c) [\delta(E - E_1) + \delta(E - E_2)] da db dc,$$

$$\bar{E} = \frac{\int_0^{\infty} g(E) dE}{\int_0^{\infty} g(E) dE}, \quad \epsilon = \frac{E}{\bar{E}}, \quad D(\epsilon) = \frac{g(\epsilon \bar{E})}{\int_{-\infty}^{\infty} g(\epsilon \bar{E}) d\epsilon}. \tag{21}$$

Once again $f(x)$ is the PDF of matrix elements. Here, $E_{1,2} = \frac{1}{2}(a + c \pm \sqrt{(a - c)^2 + 4b^2})$ for R_1 and $E_{1,2} = a \pm \sqrt{b^2 + c^2}$ for R_2 . For R_1 , $g(E)$ can be obtained from (21), by using Gaussian PDF, defining $a + c = u, a - c = v$ and crashing the delta function w.r.t. u . Next, we use polar co-ordinates $v = r \cos \theta$ and $b = \frac{r}{2} \sin \theta$ to get

$$g(E) = A'e^{-2E^2} \int_0^{\infty} \int_0^{2\pi} \cosh(2Er) e^{-r^2(\frac{7}{8} + \frac{\cos 2\theta}{8})} r dr d\theta \tag{22}$$

which reduces to a one-dimensional integral

$$g(E) = A''e^{-2E^2} \int_0^{\infty} r \cosh(2Er) e^{-7r^2/8} I_0(r^2/8) dr. \tag{23}$$

For R_2 with Gaussian PDF, we crash the delta function w.r.t. the variable a and use polar co-ordinates $b = r \cos \theta, c = r \sin \theta$ and we get a simple form

$$g(E) = e^{-E^2} [2 + \sqrt{2\pi}E \operatorname{erf}(E/\sqrt{2})e^{E^2/2}]/(4\sqrt{\pi}). \tag{24}$$

This function is normalized to 1 in for $E \in (-\infty, \infty)$, \bar{E} calculated in $E \in (0, \infty)$ is $\frac{4+\pi}{4\sqrt{\pi}} = 1.0073$, consequently, $D(\epsilon) = g(\epsilon)$. See the $D(\epsilon)$ histograms in Figure 4 for eigenvalues of $N = 8 \times 10^4$ matrices:(a) R_1 and (b) R_2 where the matrix elements are Gaussian random numbers with mean 0 and variance 1. In Figure 4, $D(\epsilon)$ ((23) and (24)) matches well with the histograms. Usually, $D(\epsilon)$ is plotted by taking $\epsilon = E/E_m$, where E_m is the maximum of the eigenvalues and $D(\epsilon)$ is studied for $-1 \leq \epsilon \leq 1$. With regard to this the x-axis could be scaled down to the domain $[-1, 1]$ to see that the ensembles of 2×2 real matrices defy the semi-circle law which is observed for real symmetric matrices of large order. We also find that $D(\epsilon)$ for both R_1 and R_2 are sensitive to the PDF of matrix elements.

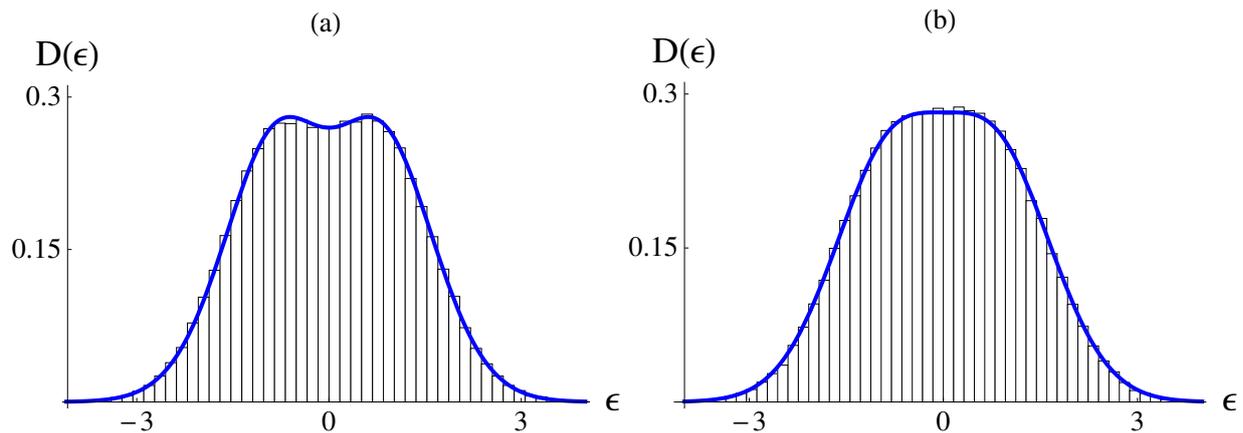


FIGURE 4. Distribution of eigenvalues $D(\epsilon)$ for R_1 (a) and R_2 (b) in (1) under Gaussian PDF of matrix elements. The solid line (blue) is due to (23) and (24). The histograms are due to an ensemble of $N = 8 \times 10^4$ matrices.

4. CONCLUSIONS

Our analytic and semi-analytic results on $P(S)$ for two modifications of 2×2 real symmetric matrices in (7), (10), (13), (15), (18) and (19) under various probability distribution functions and the plotted $p(s)$ in Figures 1–3 are new and instructive. They all give the linear level repulsion as αs near $s = 0$ but notably α is not fixed. The Maxwellian PDF ($f(0) = 0$) of matrix elements presents a striking result wherein the level repulsion near $s = 0$ is cubic. It will be further interesting to investigate spectral distributions for $n \times n$ matrices with PDFs which are non-symmetric and vanish at $x = 0$. The distribution of eigenvalues for two real matrices under Gaussian PDF obtained in (23) and (24) are also new and instructive.

ACKNOWLEDGEMENTS

S. K. wishes to thank Dr. Shashi C. L. Srivastava, VECC, Kolkata, for some clarifications on RMT.

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