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NON-UNITARY TRANSFORMATION OF QUANTUM TIME-DEPENDENT NON-HERMITIAN SYSTEMS

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Abstract. We provide a new perspective on non-Hermitian evolution in quantum mechanics by emphasizing the same method as in the Hermitian quantum evolution. We first give a precise description of the non unitary transformation and the associated evolution, and collecting the basic results around it and postulating the norm preserving. This cautionary postulate imposing that the time evolution of a non Hermitian quantum system preserves the inner products between the associated states must not be read naively. We also give an example showing that the solutions of time-dependent non Hermitian Hamiltonian systems given by a linear combination of $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ are obtained thanks to time-dependent non-unitary transformation.

Keywords: non-Hermitian quantum mechanics; time-dependent Hamiltonian systems; non-unitary time-dependent transformation.

1. Introduction

One of the postulates of quantum mechanics is that the Hamiltonian is Hermitian, as this guarantees that the eigenvalues are real. This postulate result from a set of postulates representing the minimal assumptions needed to develop the theory of quantum mechanics. One of these postulates concerns the time evolution of the state vector $|\psi(t)\rangle$ governed by the Schrödinger equation which describe how a state changes with time:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

where H is the Hamiltonian operator corresponding to the total energy of the system. The time dependent Schrödinger equation is the most general way of describing how a state changes with time. Formally, we can evolve a wavefunction forward in time by applying the time-evolution operator. For a Hamiltonian which is time independent, we have $|\psi(t)\rangle = U(t,t_0)\ |\psi(t_0)\rangle$, where $U(t,t_0) = \exp(-iH(t-t_0)/\hbar)$ denotes the time-evolution operator. The time-evolution operator is an example of a unitary operator. The latter are defined as transformations which preserve the scalar product, $\langle\psi(t)|\psi(t)\rangle = \langle\psi(t_0)|\psi(t_0)\rangle$, i.e., the norm $\langle\psi(t)|\psi(t)\rangle$ is time independent.

The study of time-dependent systems has been a growing field not only for its fundamental physical perspective but also for its applicability, such as quantum optics. There has been attracted attention of physicists in the analytical solutions of the one-dimensional Schrodinger equation with a time-dependent Hamiltonian. The origin of this development was no doubt the discovery of an exact invariant by Lewis [1, 2] and Lewis and Riesenfeld [3] which exploited the invariant operators to solve quantum-mechanical problems. The invariants method [3] is very simple due to the relationship between the eigenstates of the invariant operator and the solutions to the Schrödinger equation by means of the phases. Exploiting the invariant operator theory several authors, for instance [4–14], have studied extensively in the literature two models. One of them is the time-dependent generalized harmonic oscillator with the symmetry of the $\mathrm{SU}(1,1)$ dynamical group, the other is the two-level system possessing an $\mathrm{SU}(2)$ symmetry. In this respect, M. Maamache [15] has shown that, with the help of the appropriate time-dependent unitary transformation instead of the invariant operator, the Hamiltonian of the $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ algebra can be transformed into the time-independent Hamiltonian multiplied by an overall time-dependent factor.

The quantum mechanics is capable of working for some non-Hermitian quantum systems. However, the Hermiticity is relaxed to be pseudo-Hermiticity [16] or PT symmetry in non-Hermitian quantum mechanics, where is a linear Hermitian or an anti-linear anti-Hermitian operator, and P and T stand for the parity and time-reversal operators, respectively. The theories of non-Hermitian quantum mechanics have been developed quickly in recent decades, the reader can consulte the articles [17, 18] and references cited therein.

Systems with time-dependent non-Hermitian Hamiltonian operators have been studied in [19–35]. The most recent monograph [36] can be consulted for introduction of the non-stationary theory.

In this work, we use the same strategy as done in [15] to solve the Schrödinger equation for the time-dependent Hermitian Hamiltonian systems . We introduce the non-unitary transformation $V(t)$ mapping the solution $|\psi(t)\rangle$ of the time-dependent Schrödinger equation involving a non-Hermitian Hamiltonian $H(t)$ to a solution of the $|\phi(t)\rangle$ involving a non Hermitian Hamiltonian $\mathcal{H}(t)$ required as a product of a simple time-independent Hamiltonian H_0 and a time-dependent factor $g(t)$. After performing transformation of Schrödinger equation, the problem becomes exactly solvable but the evolution is not unitary and consequently doesn't preserve the scalar product. In order to obtain a conserved norm we postulate that the time evolution of a quantum system preserves, not just the normalization of the quantum states, but also the inner products between the associated states $\langle\phi(t)|\phi(t)\rangle = \langle\phi(0)|\phi(0)\rangle$. This is the main result of this paper.

As an illustration of our method, we present a specific quantum system given by a linear combination of $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ generators. For this we introduce, in § 2, a formalism based on the time-dependent non-unitary transformations and we show that the time-dependent non-Hermitian Hamiltonian is related to an associated time-independent Hamiltonian multiplied by an overall time-dependent factor. In § 3, we illustrate our formalism introduced in the previous section by treating a non-Hermitian $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ time-dependent quantum problem and finding the exact solution of the Schrödinger equation without making recourse to the pseudo-invariant operator theory as been done in [34, 35] or to the technique presented in [28, 29]. Finally, § 4 concludes our work.

Our analysis has shown that the key of solving the time-dependent Schrödinger equation is to find a way to transform the problem to a standard integrable form.

2. Formalism

Consider the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{2}$$

with $\hbar = 1$ and $H(t)$ is the time-dependent non-Hermitian Hamiltonian operator. Coming back to the evolution equation (2), we perform a non-unitary transformation to the wavefunction as follows:

$$|\phi(t)\rangle = V(t)|\psi(t)\rangle. \tag{3}$$

This is essentially a change of representation from $|\psi(t)\rangle$ to $|\phi(t)\rangle$ so that the evolution of the quantal system in the new representation is governed by the following evolution equation

$$i\frac{\partial}{\partial t}|\phi(t)\rangle = \mathcal{H}(t)|\phi(t)\rangle. \tag{4}$$

The operator $H(t)$ changes into $\mathcal{H}(t)$

$$\mathcal{H}(t) = V(t)H(t)V^{-1}(t) + i\frac{\partial V(t)}{\partial t}V^{-1}(t). \tag{5}$$

The evolution equation (4) shares the same form as the original evolution equation in (2). However this equation can readily be solved if we make a proper choice for the non-unitary operator $V(t)$. In this way, we are seeking a representation in which the associated evolution equation can be solved easily. This is done by employing the following criterion: $\mathcal{H}(t)$ governing the evolution of (4) is required to be in the form

$$\mathcal{H}(t) = g(t)H_0. \tag{6}$$

The implication of these results is clear, the transformed Hamiltonian is a product of a simple time-independent Hamiltonian H_0 and a time-dependent factor $g(t)$. Consequently, the original time-dependent non-Hermitian quantum problem is completely solved.

If we now define $|\zeta_n\rangle$ eigenstate of H_0 with a constant eigenvalue λ_n, we can write the eigenvalue equation in the form

$$H_0|\zeta_n\rangle = \lambda_n|\zeta_n\rangle. \tag{7}$$

As is easily verified, the solution $|\phi_n(t)\rangle$ of the Schrödinger equation (4) can be written as

$$|\phi_n(t)\rangle = \exp\left(i\lambda_n\int_0^t g(t')dt'\right)|\zeta_n\rangle. \tag{8}$$

Of note it follows immediately that the time evolution of a quantum system described by $|\phi_n(t)\rangle$ doesn't preserve the normalization i.e., the inner product of evolved states $|\phi_n(t)\rangle$ depend on time:

$$\langle\phi_n(t)|\phi_n(t)\rangle = \exp\operatorname{Im}\left(\lambda_n\int_0^t g(t'))dt'\right)\langle\phi_n(0)|\phi_n(0)\rangle. \tag{9}$$

At this stage, we will postulate, like the Hermitian case, that the time evolution of a quantum system preserves, not just the normalization of the quantum states, but also the inner products between the associated states.

Postulate: The time evolution of a non-Hermitian quantum system preserves the normalization of the associated ket.

The preservation of the norm of the state is associated with conservation of probability $\langle\phi_n(t)|\phi_n(t)\rangle = \langle\phi_n(0)|\phi_n(0)\rangle$, implying that the imaginary part of the phase vanishes, which imposes that $g(t)$ and λ_n are reals and consequently the Hamiltonian $\mathcal{H}(t)$ should be Hermitian. The important implication of this results is clear. The requirement of the stationarity of the scalar product between the states $|\phi_n(t)\rangle$ implies that the operator $\mathcal{H}(t)$ is diagonal in the basis $\{|\zeta_n\rangle\}$ with eigenvalue $\exp\bigl(i\lambda_n\int_0^t g(t')dt'\bigr)$.

In the original representation $|\psi_n(t)\rangle$, the solution to the evolution equation in (2) will then be given by

$$|\psi_n(t)\rangle = \exp\left(i\lambda_n\int_0^t g(t')dt'\right)V^{-1}(t)|\zeta_n\rangle. \tag{10}$$

As we are only changing our description of the system by changing basis, we must preserve the inner product between vectors. Explicitly, from preserving of this inner product between states $|\phi_n(t)\rangle$, we can now define the inner product between states $|\psi_n(t)\rangle = V^{-1}(t)|\phi_n(t)\rangle$ as

$$\langle\psi_n(t)|V^+(t)V(t)|\psi_n(t)\rangle = \langle\psi_n(0)|V^+(0)V(0)|\psi_n(0)\rangle \tag{11}$$

which has both a positive definite signature and leaves the norms of vectors stationary in time.

3. Application: SU(1, 1) and SU(2) non-Hermitian time-dependent systems

However, if the Hamiltonian $H(t)$ takes the following form:

$$H(t) = 2\omega(t)K_0 + 2\alpha(t)K_- + 2\beta(t)K_+, \tag{12}$$

where $(\omega(t), \alpha(t), \beta(t)) \in C$ are arbitrary functions of time. The Hermitian operator K_0 and $K_+ = (K_-)^+$ forms a closed Lie algebra. In this paper we shall concentrate on a particular time-dependent non-Hermitian Hamiltonian *(12)* which comprises SU(1, 1) and SU(2) group generators, where K_0, K_- and K_+ form the SU(1, 1) and SU(2) Lie algebra written in the following unified form:

$$\begin{cases} [K_0, K_+] = K_+, \\ [K_0, K_-] = -K_-, \\ [K_+, K_-] = DK_0 \end{cases} \tag{13}$$

The Lie algebra of SU(1, 1) and SU(2) corresponds to $D = -2$ and 2 in the commutation relations (13), respectively. K_+ is the creation operator and K_- is the annihilation operator when acting on eigenfunctions of K_0.

Then the non-unitary transformation operator $V(t)$ can be expressed locally in the following form

$$V(t) = \exp[2\varepsilon(t)K_0 + 2\mu(t)K_- + 2\mu^*(t)K_+], \tag{14}$$

where ε , μ are arbitrary real and complex time-dependent parameters respectively. We shall disentangle this exponential operator into a product of exponential operators [37, 38]. This procedure provides a way to uncouple exponential operators which are not necessarily unitary. Now we have

$$V(t) = e^{\vartheta_+(t)K_+}e^{\ln\vartheta_0(t)K_0}e^{\vartheta_-(t)K_-}. \tag{15}$$

We chose this particular form (15) for $V(t)$ because it is expressed as a product of exponential operators, and direct differentiation with respect to time for this operator can be readily carried out. The time dependent coefficients $\vartheta_0(t)$ and $\vartheta_\pm(t)$ read

$$\begin{aligned} \vartheta_0(t) &= \left(\cosh\theta - \frac{\varepsilon}{\theta}\sinh\theta\right)^{-2}, & \theta &= \sqrt{\varepsilon^2 + 2D|\mu|^2}, \\ \vartheta_+(t) &= \frac{2\mu^*\sinh\theta}{\theta\cosh\theta - \varepsilon\sinh\theta}, & \vartheta_-(t) &= \frac{2\mu\sinh\theta}{\theta\cosh\theta - \varepsilon\sinh\theta}. \end{aligned} \tag{16}$$

The notation may be simplified even further by introducing some new quantities [29]

$$z = \frac{2\mu}{\varepsilon} = |z|e^{i\varphi}, \qquad \phi = \frac{|z|}{1 - \frac{\varepsilon}{\theta}\coth\theta}, \qquad \chi(t) = -\frac{\cosh\theta + \frac{\varepsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{\varepsilon}{\theta}\sinh\theta}. \tag{17}$$

With this adopted notation, the coefficients in (16) simplify to

$$\vartheta_\pm = -\phi e^{\mp i\varphi}, \qquad \vartheta_0 = -\frac{D}{2}\phi^2 - \chi\ . \tag{18}$$

Using the relations

$$\begin{cases}\exp[\vartheta_- K_-]K_0\exp[-\vartheta_- K_-] = K_0 + \vartheta_- K_-,\\ \exp[\vartheta_+ K_+]K_0\exp[-\vartheta_+ K_+] = K_0 - \vartheta_+ K_+,\end{cases} \tag{19}$$

$$\begin{cases}\exp[\ln\vartheta_0 K_0]K_-\exp[-\ln\vartheta_0 K_0] = \frac{K_-}{\vartheta_0},\\ \exp[\vartheta_+ K_+]K_-\exp[-\vartheta_+ K_+] = K_- + D\vartheta_+ K_0 - \frac{D}{2}\vartheta_+^2 K_+,\end{cases} \tag{20}$$

$$\begin{cases}\exp[\ln\vartheta_0 K_0]K_+\exp[-\ln\vartheta_0 K_0] = \vartheta_0 K_+,\\ \exp[\vartheta_- K_-]K_+\exp[\vartheta_- K_-] = K_+ - D\vartheta_- K_0 - \frac{D}{2}\vartheta_-^2 K_-,\end{cases} \tag{21}$$

$$i\frac{\partial V(t)}{\partial t}V^{-1}(t) = \frac{i}{\vartheta_0}\Big((\dot\vartheta_0 + D\vartheta_+\dot\vartheta_-)K_0 + \dot\vartheta_- K_- + \Big(\vartheta_0\dot\vartheta_+ - \vartheta_+\dot\vartheta_0 - \frac{D}{2}\vartheta_+^2\dot\vartheta_-\Big)\Big) \tag{22}$$

and putting them into (5), we obtain, after some algebra the transformed Hamiltonian

$$\mathcal{H}(t) = 2\mathcal{W}(t)K_0 + 2\mathcal{Q}(t)K_- + 2\mathcal{Y}(t)K_+, \tag{23}$$

where the coefficient functions are

$$\mathcal{W}(t) = \frac{1}{\vartheta_0}\Big(\omega\Big(\frac{D}{2}\vartheta_+\vartheta_- - \chi\Big) + D(\vartheta_+\alpha + \vartheta_-\beta\chi) + \frac{i}{2}(\dot\vartheta_0 + D\vartheta_+\dot\vartheta_-)\Big), \tag{24}$$

$$\mathcal{Q}(t) = \frac{1}{\vartheta_0}\Big(\omega\vartheta_- + \alpha - \frac{D}{2}\beta\vartheta_-^2 + i\frac{\dot\vartheta_-}{2}\Big), \tag{25}$$

$$\mathcal{Y}(t) = \frac{1}{\vartheta_0}\Big(\omega\chi\vartheta_+ - \frac{D}{2}\alpha\vartheta_+^2 + \beta\chi^2 + \frac{i}{2}\Big(\vartheta_0\dot\vartheta_+ - \vartheta_+\dot\vartheta_0 - \frac{D}{2}\vartheta_+^2\dot\vartheta_-\Big)\Big). \tag{26}$$

The evolution equation (4) under $\mathcal{H}(t)$ can readily be solved if we make a proper choice for the non-unitary operator $V(t)$. In this way, we are seeking a representation in which the associated evolution equation can be solved easily. This is done by employing the following criterion: $\mathcal{H}(t)$ in (23) is required to be diagonal in the eigenbasis $\{|\zeta_n\rangle\}$ of K_0. The above requirement can be achieved if and only if the inner product between the associated states $|\phi_n(t)\rangle$ is preserved, which is achieved by imposing $\mathcal{Q}(t) = 0$, $\mathcal{Y}(t) = 0$ and $\operatorname{Im}\mathcal{W}(t) = 0$. These conditions lead, by using (18) and after some algebra, to the following constraints

$$\dot\varphi = 2|\omega|\cos\varphi_\omega - 2\frac{|\alpha|}{\phi}\cos(\varphi_\alpha - \varphi) + D\phi|\beta|\cos(\varphi + \varphi_\beta), \tag{27}$$

$$\dot\phi = -2\phi|\omega|\sin\varphi_\omega + 2|\alpha|\sin(\varphi_\alpha - \varphi) - D\phi^2|\beta|\sin(\varphi + \varphi_\beta), \tag{28}$$

$$\dot\vartheta_0 = \frac{2\vartheta_0}{\phi}\big(-2\phi|\omega|\sin\varphi_\omega + |\alpha|\sin(\varphi_\alpha - \varphi) + (\chi - D\phi^2)|\beta|\sin(\varphi + \varphi_\beta)\big), \tag{29}$$

by which ϑ_-, ϑ_+ and ϑ_0 are detemined for given values of $\omega(t)$, $\alpha(t)$ and $\beta(t)$. It is important to note here that when considering the time-dependent coefficient μ to be real function instead of complex one, i.e., the polar angles φ vanish, the auxiliary equations (27)–(29) that appear automatically in this process are identical to equations (28)-(30) for Maamache et al [35] who used the general method of Lewis and Riesenfield to derive them. Then the transformed Hamiltonian $\mathcal{H}(t)$ becomes

$$\begin{aligned}\mathcal{H}(t) &= 2\operatorname{Re}(\mathcal{W}(t))K_0\\ \operatorname{Re}(\mathcal{W}(t)) &= \big(|\omega|\cos\varphi_\omega + D\phi|\beta|\cos(\varphi + \varphi_\beta)\big).\end{aligned} \tag{30}$$

The implication of the results is clear. The original time-dependent quantum-mechanical problem posed through the Hamiltonian (12) is completely solved if the wave function for the related transformed Hamiltonian $\mathcal{H}(t)$ defined in (8) is obtained. The exact solution of the original equation (2) can now be found by combining the above results. We finally obtain

$$|\psi_n(t)\rangle = \exp\Big(i\lambda_n\int_0^t 2\big(|\omega|\cos\varphi_\omega + D\phi|\beta|\cos(\varphi + \varphi_\beta)\big)dt'\Big)V^{-1}(t)|\zeta_n\rangle. \tag{31}$$

Now, we consider the $\mathrm{SU}(1,1)$ case first where $D = -2$. The $\mathrm{SU}(1,1)$ Lie algebra has a realization in terms of boson creation and annihilation operators a^+ and a such that

$$K_0 = \frac{1}{2}\Big(a^+a + \frac{1}{2}\Big), \qquad K_- = \frac{1}{2}a^2, \qquad K_+ = \frac{1}{2}a^{+2}. \tag{32}$$

Then, the Hamiltonian (12) describes the generalized time dependent Sawson Hamiltonian [29]. If $\omega(t)$, $\alpha(t)$ and $\beta(t)$ are reals constant, this Hamiltonian has been studied extensively in the literature by several authors, for instance [39–46]. Substitution of $D = -2$, and $\lambda_n = \frac{1}{2}(n + \frac{1}{2})$ into (31) yields

$$|\psi_n(t)\rangle = \exp\left(i\left(n + \frac{1}{2}\right)\int_0^t \big(|\omega|\cos\varphi_\omega - 2\phi|\beta|\cos(\varphi + \varphi_\beta)\big)dt'\right)V^{-1}(t)|n\rangle, \tag{33}$$

where $|\zeta_n\rangle = |n\rangle$ are the eigenvectors of K_0.

For $D = 2$, Hamiltonian (12) possesses the symmetry of the dynamical group SU(2). A spin in a complex time-varying magnetic field is a practical example in this case [47–53]. Let $K_0 = J_z$ and $K_\mp = J_\mp$. $|\zeta_n\rangle = |j,n\rangle$ are the eigenvectors of J_z, i.e., $J_z\ |j,n\rangle = n|j,n\rangle$. The next step is the calculation of the solutions (31) which are given by

$$|\psi_n(t)\rangle = \exp\left(in\int_0^t \big(|\omega|\cos\varphi_\omega + 2\phi|\beta|\cos(\varphi + \varphi_\beta)\big)dt'\right)V^{-1}(t)|j,n\rangle, \tag{34}$$

4. Conclusion

It has been established [21–23] that the general frame-work for a description of unitary time evolution for time-dependent non-Hermitian Hamiltonians can be based on the use of a time-dependent metric operator. The unitarity of the time evolution can be guaranteed but the Hamiltonian (the generator of the Schrödinger time-evolution) must remain unobservable in general. The latter results were recently illustrated in [28, 29]. In this present work, we adapted another approach based on a time-dependent non-unitary transformation of time-dependent Hermitian Hamiltonians [15] to solve the Schrödinger equation for the time-dependent non-Hermitian Hamiltonian. Starting with the original time-dependent non-Hermitian Hamiltonian $H(t)$ and through a non-unitary transformation $V(t)$ we derive the transformed $\mathcal{H}(t)$ as time independent Hamiltonian multiplied by a time-dependent factor. Then, we postulate that the time evolution of a non Hermitian quantum system preserves the inner products between the associated states, which allows us to identify this transformed Hamiltonian $\mathcal{H}(t) = 2\,\mathrm{Re}(\mathcal{W}(t))K_0$ as Hermitian. Thus, our problem is completely solved.

Evidently, we then have presented to illustrate this theory: the $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ non-Hermitian time-dependent systems described by the Hamiltonian (12) when applying the non-unitray transformation $V(t)$ we obtain the transformed Hamiltonian $\mathcal{H}(t)$ as linear combination of K_0 and $K_\mp$. Consequently, we must disregard the prefactors of the operators $K_\mp$. To this end, we next require that the coefficients $\mathcal{Q}(t) = 0$ and $\mathcal{Y}(t) = 0$ defined in (25)–(26). Then, by using the postulat that the inner products between the associated states is preserved allows us to require that $\mathrm{Im}\,\mathcal{W}(t) = 0$ and to identify the transformed Hamiltonian $\mathcal{H}(t) = 2\,\mathrm{Re}(\mathcal{W}(t))K_0$ as Hermitian.

The $\mathrm{SU}(1,1)$ example provided was previously solved in [29] with the requirement $\mathcal{Q}(t) = \mathcal{Y}^+(t)$. At first glance, this means our new solution is just a special case of the one provided in [29] . In fact, this is not the case because the solution of Shrodinger equation can never be obtained using uniquely this requirement; i.e.$\mathcal{Q}(t) = \mathcal{Y}^+(t)$. In order, to solve the Shrodinger equation associated to their time dependent Hermitian Hamiltonian obtained by the requirement $\mathcal{Q}(t) = \mathcal{Y}^+(t)$, the authors of [29] have adapted the Lewis and Riesenfeld time-dependent invariants technique. Thus, they use two steps to solve the generalized Swanson Hamiltonian. However, our method is straightforward to obtain the solution of the generalized Swanson Hamiltonian and the Lewis and Riesenfeld time-dependent invariant is a consequence.

Finaly, we also found the exact solutions of the generalized Swanson model and a spinning particle in a time-varying complex magnetic field.

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