Common Mathematical Model of Fatigue Characteristics

Z. Maléř, S. Slavík, T. Marczi, M. Růžička

This paper presents a new common mathematical model which is able to describe fatigue characteristics in the whole necessary range by one equation only:

\[ \log N = A(R) + B(R) \cdot \log S_a \]

where \( A(R) = AR^2 + BR + C \) and \( B(R) = DR^2 + AR + F \).

This model was verified by five sets of fatigue data taken from the literature and by our own three additional original fatigue sets. The fatigue data usually described the region of \( N = 10^4 \) to \( 3 \times 10^8 \) and stress ratio of \( R = -2 \) to 0.5. In all these cases the proposed model described fatigue results with small scatter. Studying this model, following knowledge was obtained:

- the parameter "stress ratio R" was a good physical characteristic
- the proposed model provided a good description of the eight collections of fatigue test results by one equation only
- the scatter of the results through the whole scope is only a little greater than that round the individual S/N curve
- using this model while testing may reduce the number of test samples and shorten the test time
- as the proposed model represents a common form of the S/N curve, it may be used for processing uniform objective fatigue life results, which may enable mutual comparison of fatigue characteristics.

Keywords: fatigue characteristics, mathematical model, stress ratio R.

1 Introduction

It is a great advantage if the designer has reliable fatigue characteristics of the main carrying parts while designing a new aircraft. The first complete characteristics of wings were make in internationally known laboratories. Nowadays, major producers create the characteristics of their own specific, commonly used parts. This paper offers a new mathematical model that enables fatigue tests to be considered during testing, and in this way fatigue characteristics can be established very reliably and economically. For illustration, three examples of the use of this model are shown.

2 The present state of fatigue characteristics expression

\( S_a \), amplitude and \( S_m \), mean stress of the load cycle are decisive and at the same time, also geometrically transparent factors influencing the life of structure elements, the influence of these quantities on the life, expressed by the number of cycles until failure \( A \), is often presented graphically for the system of S/N curves [1] (see Fig. 1) or in the form of the Haigh diagram [2] (see Fig. 4).

Where computers were introduced, the graphical representation of fatigue characteristics was no longer satisfactory, and mathematical ways of expression were searched. An expression used to describe the fatigue characteristics of wing and horizontal tail surfaces [1, 3] was one of the first models

\[ \log N = a(S_m) + b(S_m) \log(S_a - 0.835) \] (1)

where \( a(S_m) \) and \( b(S_m) \) represent the positions (intercept on log N axis) and slopes of S/N curves left branches.

NOTE: This paper will not deal with the right side of S/N curves, i.e., fatigue limits.

Parameters \( a(S_m) \) and \( b(S_m) \) are functionally dependent on the mean stress value (see Fig. 1) and they are described in four segments by means of polynomials of the first and third degree.

This way of expression is accurate, and it enables a computer to be used. The width of the segments and the mathematical function are chosen to replace the results of fatigue tests as accurately as possible. However, selection of the segment width and suitable polynomials leads to the loss of the general character. The system of polynomials accurately describes the geometrical form of the results but this description has no the physical sense. The system does not describe the physical relations among \( S_a, S_m \) and \( N \).

After supplying the existing results with another group of results, or when processing another set of test results it is necessary to choose new width segments and a new system of polynomials to achieve an accurate presentation of the test results. Section widths and polynomials are chosen by way of trial.

For these reasons this model is not generally valid, nor can it be used in fatigue test design or during fatigue tests to control them. It is necessary to find a new mathematical model.

The requirements for the new mathematical model can be expressed as follows:

1. The model should have the basic form of the common model of S/N curve

\[ \log N = a + b \log(S_a) \]

2. Functions \( a \) and \( b \) should always be the same for any tested structure or element within the whole study range, and only its parameters may vary.

3. Changes of equation parameters must depend upon test results only. If the test results are homogeneous and there is no deflection from the physical limits, the equation parameters cannot change considerably when increasing the number of test samples.

4. Functions \( a \) and \( b \) should be valid within a high-cycle area, i.e., for \( N > 10^4 \).
3 Design of the new mathematical model

When designing the model, we start from the following knowledge. It was found out, see e.g. [5], that when describing the influence of the mean (or better to say, at description of load cycle position influence) on crack growth, stress ratio \( R \) is a more suitable parameter than the mean stress \( S_m \). For this reason parameter \( R = S_{min}/S_{max} \) was used instead of \( S_m \).

The relation among \( S_a, R, N \) is described in the shape
\[
Y = \log N = A(R) + B(R) \cdot \log S_a .
\] (3a)

This function describes the system/family of \( S/N \) curve branches that correspond to different values of \( R = \text{const.} \) in the area of high-cycle fatigue, see Fig. 2. The position and the slope parameters of \( S/N \) curve branches, in co-ordinate system \( \log S_a - \log N \), depends on the value of \( R \). Dependencies \( A(R) = f(R) \) and \( B(R) = f(R) \) are expressed by means of the following functions:
\[
A(R) = AR^2 + BR + C ,
\] (3b)
\[
B(R) = DR^2 + ER + F .
\] (3c)
4 Examples of judgement of validity and accuracy of the $S_a$–$R$–$N$ model

Suggested $S_a$–$R$–$N$ model (3) was judged on five sets of fatigue results taken from the literature, and was used for processing our own threesets of test results. Validity and accuracy were judged by:

- comparing the origin and $S_a$–$R$–$N$ diagrams,
- comparing $S/N$ curve parameters ($A$,$B$,$C$) and model functions $A(R)$, $B(R)$,
- value of standard deviation $s_F$ (6).

This paper will describe the following three results.

4.1 Example 1

Fig. 1 shows $S/N$ curves of complete wings and tail planes. The test results there were taken as points read from the original curves. The dashed lines are the original $S/N$ curves according to the Data Sheets – Fatigue E.02.01 [1], and the solid lines are calculated by the new $S_a$–$R$–$N$ model, see equation (3). As we can see, the differences are negligible. The equation is:

$$\log N = A(R) + B(R) \cdot \log S_a$$

where

- $A(R) = -1.726325 R^2 - 1.643550 R + 13.560228$
- $B(R) = 0.827044 R^2 + 0.296186 R - 4.987437$.

The differences between the original and the model curves are expressed by the value of the standard deviation $s_F = 0.086$.

4.2 Example 2

Fig. 4 compares the original constant life curves (thin curves) of the MUSTANG wings and the curves established by the new $S_a$–$R$–$N$ model (3) (thick curves) [2]. The model curves were calculated in the range of $N = 10^4$ to $10^7$ cycles. The test results were taken as read from the original $N$= const. curves for $R = -1; -0.8; -0.6; \ldots ; 0.6$.

$$\log N = A(R) + B(R) \cdot \log S_a$$

where

- $A(R) = 0.542293 R^2 + 1.125954 R + 11.914455$
- $B(R) = 0.853812 R^2 - 1.948959 R - 5.884825$.

The $s_F$ value represents the differences/scatter between the new model and the original curves.

The value of the standard deviation is $s_F = 0.107$.

4.3 Example 3

Fig. 3 there are shows our original test results for the test specimen that represents a steel attachment critical point. The specimens are made of Czech Poldi L-ROL.7 steel heat treated to $R_{m_{\text{min}}} = 1080$ MPa which is similar to U.S. 4130 alloy.
The specimens were axially loaded at $R = -1; -0.5; 0.23; 0.42$. The equation describing the $S_a-R-N$ model is:

$$\log N = A(R) + B(R) \cdot \log S_a$$

where

$$A(R) = 4.930716 R^2 + 3.479890 R + 13.980960$$

$$B(R) = -2.289047 R^2 - 1.950201 R - 3.876255.$$  

The standard deviation of the test results round the $S_a-R-N$ model is $s_F = 0.167$.

Fig. 3: The original test results for the test specimen that represents a steel attachment critical point.

Fig. 4: The comparison of the original constant life curves (thin curves) of the MUSTANG wings and the curves established by the new $S_a-R-N$ model.
5 Partial conclusion

This paper presents three of the eight test groups on which the new \( S_a-R-N \) model has been tested. In all five other checked fatigue sets the new model describes the test results with similar precision and in the whole test range by one equation of (3) only.

6 Conclusion

The following conclusions can be drawn from a comparison of the original characteristics and the characteristics obtained when using the mathematical model (3):

1. The designed model was able to describe, with sufficient accuracy and reliability, the results for five different structures and elements within the range of \( N = 10^4 \) through \( 10^6 \) to \( 10^7 \), and within the range of \( R = -1 \) through 0.6, pre-processed in some way (graphical or mathematical).

2. When processing the original results of the fatigue tests, the \( S/N \) curves determined through model function (3) were nearly identical with partial \( S/N \) curves calculated for different values of \( R = \text{const} \). The scatter of test results calculated from the whole set was not much higher than that around the individual \( S/N \) curves.

3. Parameter \( R \) (together with \( S_a \)) proved to be a suitable physical parameter for describing the load cycle.

Further knowledge follows from this:

4. The \( S_a-R-N \) model can be considered as a suitable, generally valid mathematical description of fatigue characteristics within the high-cycle fatigue area \((N > 10^5)\).

5. For this reason, model (3) enables economical design of fatigue tests, because during tests not just one \( S/N \) curve but the wide region s described. The number of test samples is not much greater.

6. As the designed model is only a generalized form of a commonly used \( S/N \) curve, it provides a basis for uniform, comparable, objective and accurate processing of fatigue test results.

7. Instead of \( S_a \), \( S_{\max} \) can also be used.

7 Acknowledgment

This paper is presented thanks to the Airspace Research Center at the Institute of Aerospace Engineering at the Technical University in Brno and at the Czech Technical University in Prague, Czech Republic.