Realization of Logical Circuits with Majority Logical Function as Symmetrical Function

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The paper deals with the “production” and design of symmetrical functions, particularly aimed at the design of circuits with majority elements, which lead to interesting solutions of logical structures. The solutions are presented in several examples, which show the applicability of the procedures to the design of FPGA morphology on chips.

Keywords: Shannon extension development, Hamming weight, derivation of Boolean function, symmetrical and majority function.

1 Introduction

Binary logical circuits designed with respect to Boolean symmetrical, particularly majority, output functions are certainly worth attention. The article, therefore, makes an evaluation both by controlling binary one-digit adders and by using functions interpreted by arithmetic polynomials. It also demonstrates how effectively the Shannon decomposition of the output functions can be used in designing a circuit with majority elements.

2 Boolean function

Let a Boolean function \( f : \{0,1\}^m \rightarrow \{0,1\} : \{x_1, x_2, \ldots, x_m\} \) and \( y \) be given. If we denote the set \( \{x_i\}_{i=1}^m \) of arguments \( x_j \) by the symbol \( X \), we can briefly write \( f(X) \) instead of \( f(x_1, x_2, \ldots, x_m) \). In addition, instead of \( f(x_1, x_2, \ldots, x_{m-1}, x_{i+1}, \ldots, x_m) \), in which \( \sigma_i \in \{0,1\} \), let us simply write \( f(x=\sigma_i) \). Let \( x^\sigma = x_1 \sigma_1 \oplus x_2 \sigma_2 \oplus \ldots \oplus x_i \sigma_i \oplus \ldots \oplus x_m \). Any Boolean function \( f(x_1, x_2, \ldots, x_m) \) can be expressed, without loss of generality, by the Shannon extension development

\[
\begin{align*}
\hat{f}(x) &= \sum_{\{\sigma_1, \sigma_2, \ldots, \sigma_m\}} f(\sigma_1, \sigma_2, \ldots, \sigma_m) x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_i^{\sigma_i} \ldots x_m^{\sigma_m},
\end{align*}
\]

where \( n \leq m \) esp.

\[
\begin{align*}
f(x) &= \sum_{\{\sigma_i\}} f(x_i = \sigma_i) = \sum_{\sigma_i} f(x_i = 0) \lor x_i f(x_i = 1)
\end{align*}
\]

the functions \( f(\sigma_1, \sigma_2, \ldots, \sigma_n, x_{n+1}, \ldots, x_m) \) will be called remainder functions.

By the Hamming weight \( w_H \) \( f(X) \) of the function \( f(X) \) we understand the value of the arithmetic formula

\[
\begin{align*}
w_H(f) &= \sum f(\sigma_1, \sigma_2, \ldots, \sigma_m)
\end{align*}
\]

The partial derivation \( \frac{\partial f(X)}{\partial x_i} \) \([1]\) of function \( f(X) \) by the argument \( x \) will be termed the Boolean function

\[
\begin{align*}
\frac{\partial f(X)}{\partial x_i} &= f(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m) \oplus \\
&\quad \oplus f(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_m)
\end{align*}
\]

defining the conditions under which \( f(X) \) changes its value while the value of the argument \( x \) is changed.

For example, for \( y = x_2 x_3 \lor x_1 x_2 x_3 \), when

\[
\begin{align*}
\frac{\partial y}{\partial x_3} &= \frac{\partial y}{\partial x_2} \lor (x_2 = x_3) = \frac{\partial y}{\partial x_2}(x_2 = x_3) = 1
\end{align*}
\]

the function \( y \) changes its value while the value of the argument \( x_2 \) is changed under one condition for \( \frac{\partial y}{\partial x_2} = 1 \), which is: \( x_2 = x_3 = 1 \).

3 Boolean formulae and arithmetic polynomials

Let us have

\[
\begin{align*}
f(x_1, x_2, \ldots, x_m) &= \lor_{\sigma_i} x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_i^{\sigma_i} \ldots x_m^{\sigma_m},
\end{align*}
\]

where \( k = 1, 2, \ldots, m \) and \( i = 0,1, \ldots, 2^{m-1} \), the function \( f(X) \) is represented by a normal disjunctive formula \( (nfdf f) \). If there holds

\[
\begin{align*}
\left( x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_i^{\sigma_i} \ldots \right) \left( x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_i^{\sigma_i} \ldots \right) = 0,
\end{align*}
\]

where \( k = 1, 2, \ldots, m \) and \( i = 0,1, \ldots, 2^{m-1} \), the conjuncts presented are termed orthogonal, i.e., all conjuncts of a complete normal disjunctive formula of the symmetrical Boolean function (see Paragraph 4) are mutually orthogonal. If all conjuncts \( nfdf f \) are mutually orthogonal, we can also write

\[
\begin{align*}
f(x_1, x_2, \ldots, x_m) &= \lor_{\sigma_i} x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_i^{\sigma_i} \ldots x_m^{\sigma_m},
\end{align*}
\]

Note that the function \( f(X) \) can also be conveniently expressed by the Boolean (Zegalkin) polynomial \([4]\).

Since, as can easily be confirmed, the following equality holds:

\[
\begin{align*}
x \lor y = x + y - xy
\end{align*}
\]

(probability addition)

\[
\begin{align*}
x y = xy
\end{align*}
\]

\[
\begin{align*}
x = x \lor \overline{x} = 1 - x
\end{align*}
\]

and also (\( x \) and \( \overline{x} \) are orthogonal)

\[
\begin{align*}
x \lor y = x \lor \overline{x} = x + (1 - x) y,
\end{align*}
\]
each $\text{ndf}(X)$ can be expressed by an arithmetic polynomial

$$f(x_1, x_2, \ldots, x_m) = a_0 + a_1 x_1 + a_2 x_2 + \ldots + a_m x_m + a_{m+1} x_1 x_2 + \ldots + a_{2m-1} x_1 x_2 x_3 + \ldots + a_{m^2} x_1 x_2 \ldots x_m = A(x_1, x_2, \ldots, x_m)$$

where $a_i \in N(i = 0, 1, \ldots, 2^m - 1)$, this can be done either by applying the equality $x \lor y = x + y - x y$, or orthogonalizing all conjuncts $\text{ndf}(X)$ and applying the absorption $x \lor \overline{x} y = x + (1 - x) y$. Note that if the Boolean function $f(X)$ is expressed by the arithmetic polynomial $A(X)$, then $f(X) = \text{sign} A(X)$.

**Example 1.** Let $\text{ndf} f(x_1, x_2, x_3) = (x_1 \lor x_2) \cdot x_3 \lor x_1 x_2$ be given. Express the given formula by means of the arithmetic polynomial $A(x_1, x_2, x_3)$, which is $xx = x^2 = x$:

- $(x_1 \lor x_2)x_3 \lor x_2 x_3 = x_1 x_3 + (x_1 \lor x_2)x_3 = x_1 x_3 + x_1 x_2 + x_2 x_3 - 2 x_1 x_2 x_3$

- $(x_1 \lor x_2)x_3 \lor x_1 x_2 = x_1 x_3 + x_1 + x_2 x_3 = x_1 x_3 + x_1 x_2 + x_2 x_3 - 2 x_1 x_2 x_3$

4 Symmetrical Boolean function

Let the bijection $X \leftrightarrow X$: 

$\{x_1, x_2, \ldots, x_m\} \leftrightarrow \{x_1, x_2, \ldots, x_m\}$ be a set of all permutations of arguments from $X$; the function $f(X)$ is called symmetrical if $f(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m)$; i.e. 0, $x \lor y$, $x y \lor x y \lor y$ is a symmetrical function. Let $\{P_j\}^k$ be a set of integers $P_j$ (called operational or characteristic numbers) such that 0 $\leq P_j \leq m$. It can be demonstrated [2] that $f(X)$ is symmetrical just if $f(a_1, a_2, \ldots, a_m) = 1$ for $w_H(a_1, a_2, \ldots, a_m) = P_j$. The symmetrical function with characteristic numbers $P_j$ will be denoted by $S_{m}^{P_j}$.

Obviously, $S_{0}^{P_j}$ = 0 and $S_{[0,1,\ldots,m]}^{P_j}$ = 1.

For example: $S_{[1]}^{x} = x \lor y$, $S_{[2]}^{x y} = x y \lor x y \lor y$.

The symmetrical function $S_{m}^{P_j}$ is elementary; for the length $|\text{ndf} S_{m}^{P_j}|$ of the completely normal disjunctive formula $S_{m}^{P_j}$ there holds $|\text{ndf} S_{m}^{P_j}| = m m_{P_j}$, since

$$\text{ndf} S_{m}^{P_j} = \bigvee_{a_{i} \in \{0,1\}} w_H(a_1, a_2, \ldots, a_m) = P_j x_1 a_1 x_2 a_2 \ldots x_m a_m.$$

There also holds [2]

$$\bigcup_{(Q \subset 2^{[0, \ldots, m]})} \{S_{m}^{P_j} (Q)\} = \bigcup_{i = 0}^{m+1} \bigcup_{i = 0}^{i + 1} \{S_{m+1}^{P_j} (Q)\} = 2^{m+1}$$

as well as

$$S_{m}^{P_j} \cdot S_{m}^{P_j} = 0$$

Any symmetrical function $S_{m}^{P_j}$ can be written in the form of $\text{ndf} S_{m}^{P_j} = \bigcup_{i = 1}^{\infty} \{P_i\}$:

$$S_{m}^{P_j} = \bigvee_{i = 1}^{\infty} \{P_i\}$$

since [2] $S_{m}^{P_j} = \bigvee_{i = 1}^{\infty} \{P_i\}$. We can also write

$$S_{m}^{P_j} = \bigvee_{i = 1}^{\infty} \{P_i\} = 1 \bigvee_{i = 1}^{\infty} \{P_i\}$$

where $r_i = \begin{cases} 1 & \text{for } i = P_j \\ 0 & \text{for } i \neq P_j \end{cases}$.

Denote the elementary symmetrical Boolean functions the representation of which in the form of normal disjunctive formulae (ndf) does not contain negated variables with the symbol $S_{n}^{m}$ ($n = 0, 1, \ldots, m$).

For example:

$$S_{[0]}^{m} = 1,$$

$$S_{[1]}^{m} = x_1 x_2 \ldots x_m,$$

$$S_{[m-1]}^{m} = x_1 x_2 \lor x_1 x_3 \ldots \lor x_{m-1} x_m,$$

$$S_{[m]}^{m} = x_1 \lor x_2 \lor \ldots \lor x_m.$$

Every function $S_{n}^{m}$ in which $n \neq m$ ($S_{n}^{m} = x_1 x_2 \ldots x_m$), can be expressed by the composition

$$S_{n}^{m} = S_{n}^{m} S_{m-n}^{m}.$$
function \( S_m^m = S_m^m[\bar{S}_m^{m-n}] \) and design, according to the constructed formulae, a structural model of the given function in one of the structurally complete systems of statistical elements [5].

Further, consider a one-digit binary half-adder or an adder (Fig. 1) with which \( a_i, b_i \) are binary augmenters, \( \Sigma_i \) is the sum in \( i \) position, and \( C_i^+ \) and \( C_i^- \) denote the transfer from the position \( i - 1 \) to the position \( i + 1 \), respectively. The half-adder can be modeled by a system of output functions \( \Sigma a_i \oplus b_i = S_{[1]}^i \) and \( C_i^+ = a_i b_i \bar{C}_i^- \) by analogy for the adder we obtain

\[
\Sigma_i = a_i \oplus b_i \oplus C_i^+ = S_{[1],3}^i \\
C_i^+ = \overline{a_i b_i} C_i^- \lor a_i b_i C_i^+ \lor a_i b_i C_i^- = S_{[2],3}^i
\]

It is therefore sufficient to provide the half-adder with an inverse disjunctor (Fig.1a) and the adder with a decoder (Fig.1b) and we obtain the products \( S_{[0],2}^i, S_{[1],2}^i, S_{[2],2}^i \), since

\[
S_{[1],2}^i = S_{[0],2}^i \bar{S}_{[1]}^2 \\
S_{[0],2}^i = S_{[0],1}^3 S_{[2],3}^i \text{ and since} \]

\[
S_{[1],3}^i \lor S_{[2],3}^i = S_{[0],2}^3 \lor S_{[0],1}^3 = S_{[0]}^3 \\
S_{[1],3}^i \lor S_{[2],3}^i = S_{[0],1}^3 \lor S_{[0],1}^3 = S_{[0],1}^3 \\
S_{[1],3}^i \lor S_{[2],3}^i = S_{[0],2}^3 \lor S_{[0],2}^3 = S_{[0],2}^3 \\
S_{[1],3}^i \lor S_{[2],3}^i = S_{[0],3}^3 \text{[3]}
\]

**Example 2:** Design a structural model with adders or half-adders modeled with a system of output symmetrical functions \( S \) – Fig. 2. Indeed,

\[
\Sigma_1 = x_1 \oplus x_2 \oplus x_3 = (x_1 \oplus x_2) \overline{x_3} \lor (x_1 = x_2) x_3 = S_{[1],3}^3 \\
C_1 = S_{[2],3}^3 \\
\Sigma_2 = x_4 \oplus S_{[3],3}^3 = x_4 S_{[3],3}^3 \lor x_4 S_{[3],3}^3 = S_{[1],3}^3 \\
C_2 = x_4 S_{[1],3}^3 = S_{[2],4}^3
\]

It is easy to obtain

\[
S_{[1],3}^3 = S_{[2],3}^3 \lor x_4 \oplus x_4 = \\
\left(S_{[0],1,3}^3 S_{[2],3}^3 \lor S_{[0],2}^3 \overline{S}_{[1],3}^3 \right) \lor \overline{x_4} \lor x_4 = \\
S_{[0],1,3}^3 S_{[2],3}^3 \lor S_{[0],2}^3 \overline{S}_{[1],3}^3 \lor S_{[0],1,3}^3 S_{[1],3}^3 \lor S_{[0],2}^3 \overline{S}_{[1],3}^3 = \\
S_{[0],1,3}^3 S_{[2],3}^3 \lor \overline{x_4} = S_{[2],3}^3
\]

\[
C_3 = S_{[2],3}^3 \lor \overline{S}_{[0],1,3}^3 \lor \overline{S}_{[0],2}^3 \lor S_{[1],3}^3 = S_{[4],3}^3 \\
= S_{[4],3}^3 \lor S_{[4],3}^3 \lor S_{[4],3}^3 = S_{[4],3}^3
\]

If the Boolean function \( f(X) \) is symmetrical, it can be suitably expressed by an arithmetic polynomial in the form

\[
f(X_1, X_2, \ldots , X_m) = b_0 + b_1 x_1 + x_2 + \ldots + x_m + \\
+ b_2 x_1 x_2 + x_3 + \ldots + x_{m-1} x_m + b_3 x_1 x_2 x_3 + x_4 x_2 + \\
+ \ldots + x_{m-2} x_{m-1} x_m + \ldots + b_{m+1} x_1 x_2 + \ldots \ldots + x_m = \\
= b_0 + b_1 x^1 + b_2 x^2 + \ldots + b_m X^m = \sum_{j=0}^{m} b_j X^j \] [6];

for example for

\[
S_{[1],2}^3 = x_1 \overline{x_2} \lor x_1 x_2 \lor x_1 x_2 = (l-x_1) x_2 + x_1 x_2 = x_1 + x_2 - x_1 x_2
\]

we obtain \( b_0 = 0, b_1 = 1, b_2 = -1 \) and \( X^0 = 1, X^1 = 1, X^2 = x_1 x_2 \).

The parametric notation \( S_m^m \) \( \{p_{i,j}\} \) being

\[
S_m^m = \sum_{j=0}^{m} b_j X^j \] [6];

\[
S_{[1],2}^3 = x_1 \overline{x_2} \lor x_1 x_2 \lor x_1 x_2 = (l-x_1) x_2 + x_1 x_2 = x_1 + x_2 - x_1 x_2
\]

we obtain \( b_0 = 0, b_1 = 1, b_2 = -1 \) and \( X^0 = 1, X^1 = 1, X^2 = x_1 x_2 \).

The parametric notation \( S_m^m \) \( \{p_{i,j}\} \) being

\[
S_m^m \{p_{i,j}\} = \sum_{j=0}^{m} b_j X^j \] [6];

\[
S_{[1],2}^3 = x_1 \overline{x_2} \lor x_1 x_2 \lor x_1 x_2 = (l-x_1) x_2 + x_1 x_2 = x_1 + x_2 - x_1 x_2
\]
Hence

\[ \sum_{i=0}^{m} b_i x^i = \sum_{p_j} \binom{m}{p_j} x^p_j (1-x)^{m-p_j} \]

we can easily (!) determine the values of coefficients \( b_i \), provided the structure of polynomials \( X_i \) is known.

**Example 3.** Construct an arithmetic polynomial of the symmetrical function \( S_{12}^3 \). Since

\[ S_{12}^3 = x_1 x_2 x_3 \vee x_1 x_2 \overline{x_3} \vee \overline{x_1} x_2 x_3 \vee \overline{x_1} x_2 \overline{x_3} \vee x_1 \overline{x_2} x_3 \vee \overline{x_1} x_2 \overline{x_3} \vee \overline{x_1} \overline{x_2} x_3 \vee \overline{x_1} \overline{x_2} \overline{x_3} = \sum_{j=1,2} \left[ \binom{3}{j} \right] X^j (1-X)^{3-j} = \left[ \binom{3}{1} \right] X (1-X)^2 + \left[ \binom{3}{2} \right] X^2 (1-X) = X - X^2 = b_0 + b_1 X^1 + b_2 X^2 + b_3 X^3. \]

Hence \( b_0 = 3, b_1 = 0, b_2 = 1, b_3 = -1 \) provided that \( X^1 = x_1 + x_2 + x_3 \) and \( X^2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \); because

\[ S_{12}^3 = (1-x_1) x_2 x_3 + x_1 (1-x_2) x_3 + x_1 x_2 (1-x_3) + (1-x_1) (1-x_2) x_3 + (1-x_1) x_2 (1-x_3) + x_1 (1-x_2) (1-x_3) = (x_1 + x_2 + x_3) - 2 (x_1 x_2 + x_1 x_3 + x_2 x_3) + 3 x_1 x_2 x_3 + (x_1 + x_2 + x_3) + (x_1 x_2 + x_1 x_3 + x_2 x_3) - 3 x_1 x_2 x_3. \]

**5 Circuits with majority elements**

Let us limit ourselves to majority elements modeled with the function \( \text{Maj}_2 \).

Since, as can be easily confirmed, there holds

\[ x \vee y = x \# y \# 1, \]
\[ x y = x \# y \# 0, \]

the \( \text{mdf} \) of the given function \( f(X) \) can be rewritten according to the above quoted equities and we can design the respective static structural model.
Example 4.: Construct a structural model given by the output function
\[ y = (x_1 x_2 \lor x_3 x_4) x_5 \]

hence
\[ y = ((x_1 \# x_2 \# 0) \# (x_3 \# x_4 \# 0)) \# 1 \# x_5 \# 0, \]

and thus Fig. 3.

It is also helpful to use the Shannon extension development of the given Boolean output function \( y = f(x) \)
\[ y = x f(x_1 = 0) \lor x f(x_1 = 1) = \]
\[ = (x \# f(x_1 = 0) \# (x \# f(x_1 = 1) \# 0) \# 0, \]

and it remains only to decide according to which argument to start and according to which arguments to continue the repeated application of the development. We, therefore, heuristically develop \( f(X) \), first according to the arguments whose change of values leads to the change of values \( f(X) \) under the highest number of conditions, i.e., at the highest.

Hamming weights pertaining to the derivation of the function \( f(X) \) according to the respective arguments.

Example 5.: Design a structural model given by the output function
\[ y = (x_1 x_2 x_3 x_4 x_5 x_6 x_7) \]

with majority elements. According to the map of the given output function, the \( ndf \) of the remainder functions of its Shannon extension development can easily be constructed (Fig. 4.).

Hence
\[ w_H \frac{\partial y}{\partial x_2} = w_H \left( y(x_2 = 0) \oplus y(x_2 = 1) \right) = \]
\[ = w_H \left( (x_1 \bar{x}_3 x_4 \lor x_1 x_3 x_5 \lor \bar{x}_2 x_3 x_4) \oplus (x_1 x_4 \lor x_1 x_3 x_5 \lor x_4 \bar{x}_3) \right) = 7, \]

When stating the formula which expresses the derivation of the function we will preferably use a map, to each field of which we will write the value in the form of a fraction:

\[ y(x_1 = 0) / y(x_1 = 1); \]

the resulting value of the remainder function formula as well as the weight of its derivation is evident (Fig. 5). There also holds

\[ w_H \frac{\partial y}{\partial x_3} = w_H \left( (x_1 x_4 \lor x_2 x_4 \lor \bar{x}_3 x_4) \oplus (x_1 x_3 \lor x_4 \lor \bar{x}_3) \right) = 8, \]

\[ w_H \frac{\partial y}{\partial x_4} = w_H \left( (x_1 x_2 x_3 x_4 x_5 x_6) \oplus (x_1 x_3 x_4 \lor x_2 x_3 x_4) \right) = 5, \]

\[ w_H \frac{\partial y}{\partial x_5} = w_H \left( (x_1 x_3 \lor x_2 x_3 x_4 x_5) \oplus x_3 x_4 \lor x_1 x_3 x_4 \right) = 7. \]

Since
\[ \max_i \left\{ w_H \frac{\partial y}{\partial x_i} \right\} = w_H \frac{\partial y}{\partial x_3} = 8, \]

we write

\[ y = x_3 \lor y(x_3 = 0) \land y(x_3 = 1), \]

where
\[ y(x_3 = 0) = \bar{x}_1 x_4 \lor x_2 x_4 \lor \bar{x}_4 x_5 \]

and
\[ y(x_3 = 1) = x_1 x_4 \lor x_1 x_3 \lor \bar{x}_2 x_3 (\text{Fig. 6}). \]

And, further, there is
\[ w_H \frac{\partial y(x_1 = 0)}{\partial x_1} = w_H \left( y(x_1 = 0, x_3 = 0) \oplus y(x_1 = 1, x_3 = 0) \right) = \]
\[ = w_H \left( x_1 x_4 \lor x_3 \oplus (x_2 x_4 \lor x_4 x_5) \right) = 2, \]
\[ \begin{align*}
\frac{\partial y(x_3 = 0)}{\partial x_2} &= w_H(y(x_2 = 0, x_3 = 0) \oplus y(x_2 = 1, x_3 = 0) = \\
&= w_H((x_1 x_4 \vee \bar{x}_2 x_3) \oplus (x_4 \vee x_5)) = 2,
\end{align*} \]
\[ \begin{align*}
\frac{\partial y(x_3 = 0)}{\partial x_4} &= w_H(y(x_3 = 0, x_4 = 0) \oplus y(x_3 = 0, x_4 = 1) = \\
&= w_H(x_5 \oplus (x_1 \vee x_2)) = 4.
\end{align*} \]

Fig. 6: Map entries of a) \( y(x_3 = 0) \), b) \( y(x_3 = 1) \) from Example 5

\[ \begin{align*}
\frac{\partial y(x_3 = 1)}{\partial x_1} &= w_H(y(x_1 = 0, x_3 = 1) \oplus y(x_1 = 1, x_3 = 1) = \\
&= w_H((x_1 x_4 \vee x_2 x_4) \oplus (x_1 \vee x_2 \vee \bar{x}_4)) = 4, \\
\frac{\partial y(x_3 = 1)}{\partial x_2} &= w_H(y(x_2 = 0, x_3 = 1) \oplus y(x_1 = 1, x_3 = 1) = \\
&= w_H((x_2 x_3 \oplus (x_2 \vee x_4 \vee \bar{x}_5)) = 5, \\
\frac{\partial y(x_3 = 1)}{\partial x_4} &= w_H(y(x_3 = 1, x_4 = 0) \oplus y(x_3 = 1, x_4 = 1) = \\
&= w_H((x_1 x_5 \vee \bar{x}_2 x_5) \oplus (x_1 \vee x_2 x_5)) = 1, \\
\frac{\partial y(x_3 = 1)}{\partial x_5} &= w_H(y(x_3 = 1, x_5 = 0) \oplus y(x_3 = 1, x_5 = 1) = \\
&= w_H(x_1 \oplus (x_2 \vee x_1 x_4)) = 3.
\end{align*} \]

Since
\[ \max_{i \in \mathbb{Z}} \left\{ w_H \frac{\partial y(x_3 = 0)}{\partial x_i} \right\} = w_H \frac{\partial y(x_3 = 0)}{\partial x_4} = 4 \]
and
\[ \max_{i \in \mathbb{Z}} \left\{ w_H \frac{\partial y(x_3 = 1)}{\partial x_i} \right\} = w_H \frac{\partial y(x_3 = 1)}{\partial x_1} = 5 \]
we write
\[ y = x_3(x_4 y(x_3 = 0, x_4 = 0) \vee x_4 y(x_3 = 0, x_4 = 1)) \]
\[ \vee x_5 y(x_1 = 0, x_3 = 1) \vee x_1 y(x_1 = 1, x_3 = 1), \]
where
\[ y(x_3 = 0, x_4 = 0) = x_5, y(x_3 = 0, x_4 = 1) = \bar{x}_1 \vee x_2, \]
\[ y(x_1 = 0, x_3 = 1) = \bar{x}_2 x_5 \text{ and} \]

Fig. 7: Structural model with majority elements from Example 5
In other words,
\[ y = \overline{x_3} \lor (x_4 \lor \overline{x_1} \lor \overline{x_2}) \lor x_3(\overline{x_1} \land x_2 \land x_5) \lor x_1(x_2 \lor x_4 \lor \overline{x_5}). \]

and hence also the structural model (Fig. 7).

Obviously, there is also
\[
y_1 \land x_2 \land x_3 = (x_1 \lor x_2) x_3 \lor x_1 x_2 = (x_1 \lor x_2) x_3 \lor x_1 x_2 = S_{x_1, x_2, x_3}^3(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3 - 2x_1 x_2 x_3.
\]

### 7 Conclusion

It appears that it is feasible to produce symmetrical Boolean functions in a sufficiently simple way by a suitable control of one-digit binary adders or by numerical representation of values of the respective arithmetic polynomials, and to design logical circuits with majority elements by applying the Shannon decomposition of the given output function through effective selection of the arguments by which the decomposition is carried.

### References


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