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A PRACTICAL WAY TO APPLY A TECHNIQUE THAT ACCELERATES TIME HISTORY ANALYSIS OF STRUCTURES UNDER DIGITISED EXCITATIONS

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Abstract. Time history analysis using direct time integration is a versatile and widely accepted tool for analysing the dynamic behaviour of structures. In 2008, a technique was proposed to accelerate the time history analysis of structural systems subjected to digitised excitations. Recently, this technique has been named as the SEB THAAT* (Step-Enlargement-Based Time-History-Analysis-Acceleration-Technique), and the determination of appropriate values for its parameter is introduced as the main challenge. To overcome this challenge, a procedure is proposed in this paper. The basis of the procedure is the comments on accuracy control in structural dynamics and numerical analysis of ordinary differential equations, legalised in the New Zealand Seismic Code, NZS 1170.5:2004. As the main achievement, by using the proposed procedure, we can apply the SEB THAAT and carry out the time history analysis clearly and with less parameter setting compared to the ordinary time history analysis. The proposed procedure is always applicable and, except when the behaviour is very complex, oscillatory and non-linear, the reductions in analysis run-time are considerable while the changes in accuracy are negligible. The performance can be sensitive to the problem, the integration method, the target response, and the severity of the non-linear behaviour. Compared to the previous tests on the SEB THAAT, the efficiency of applying the SEB THAAT using the proposed procedure is better, the sensitivity of the performance to the problem is lower, and a measure of accuracy is available. Compared to other techniques for accelerating structural dynamic analyses, the use of the SEB THAAT according to the proposed procedure has several positive points, including the simplicity of implementation.

Keywords: Structural dynamics, time integration, analysis run-time, response accuracy, digitised excitation, the SEB THAAT, clear application, NZS 1170.5:2004.

1. Introduction

In many structural analyses, the behaviour is dynamic and non-linear. The semi-discretised models can be expressed as [1–6]:

$$\begin{aligned}
&\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}_{\text{int}}(t) = \mathbf{f}(t), \qquad 0 \le t \le t_{\text{end}}, \\
&\text{Initial conditions:} \left| \begin{array}{l} \mathbf{u}\,(t=0) = \mathbf{u}_0, \\ \dot{\mathbf{u}}\,(t=0) = \dot{\mathbf{u}}_0, \\ \mathbf{f}_{\text{int}}\,(t=0) = \mathbf{f}_{\text{int}_0}, \end{array} \right. \\
&\text{non-linearity constraints: } \mathbf{Q},
\end{aligned} \tag{1}$$

where

t is the time,

t_{end} stands for the analysis interval,

$\mathbf{M}$ is the mass matrix,

$\mathbf{f}_{\text{int}}$ indicates the vector of internal force,

$\mathbf{f}(t)$ indicates the vector of excitation (external force),

$\mathbf{u}(t)$ denotes the vector of displacement,

$\dot{\mathbf{u}}(t)$ denotes the vector of velocity,

$\ddot{\mathbf{u}}(t)$ denotes the vector of acceleration,

$\mathbf{u}_0$ implies the initial displacement,

$\dot{\mathbf{u}}_0$ implies the initial velocity,

$\mathbf{f}_{\text{int}_0}$ implies the initial internal force ($\mathbf{f}_{\text{int}_0}$, is not needed in linear problems, it may be essential in the presence of material non-linearity),

$\mathbf{Q}$ represents the constraints that distinguish non-linear behaviour from linear behaviour, e.g. rigid barriers cannot be passed (see e.g. [7]).

For the analysis of Equation (1), direct time integration (see Figure 1) is a versatile tool that generally leads to approximate solutions after lengthy computations [1, 2]. Accuracy and run-time are very important in advanced computations, where many and/or lengthy analyses are to be carried out, e.g. IDA (Incremental Dynamic Analysis) and analytical fragility curve computations [8–10]. Accordingly, considerable effort is being made to increase the accuracy and/or to reduce the analysis run-time. Some of the main approaches are:

(1.) reducing the structural models by replacing them with models with fewer degrees of freedom [11–13],

(2.) reducing the number of excitation records, in applications such as seismic analysis [14–16],

(3.) reducing the number of oscillatory modes [17–19],

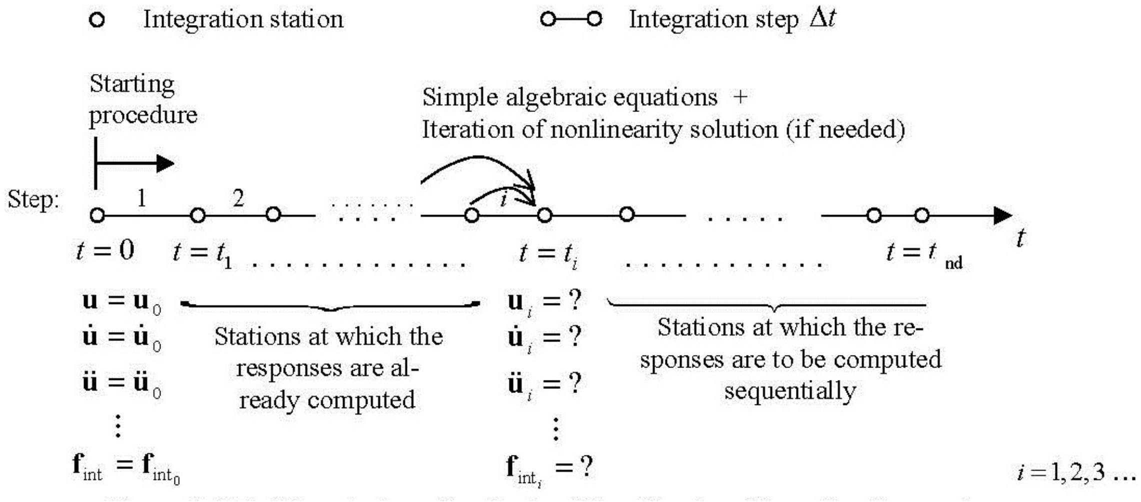


FIGURE 1. Brief description of analysis of Equation (1) using direct time integration.

(4.) time parallel methods [20–22],

(5.) using higher order/more accurate analysis methods/strategies, e.g. see [2, 23–27].

Meanwhile, in the last decades, direct down sampling, truncation methods, and their combinations, are used for faster analyses of structures under digitised excitations [28–32].

Returning to Equation (1) and the time integration computation, when $\mathbf{f}(t)$ is available in a digitised format, the generally accepted comment for the integration step is as follows [1, 33–36]:

$$\Delta t = \min\left(\frac{T}{\chi},\ \Delta t_{\text{cr}},\ \Delta t_{\text{CFL}},\ {}_f\Delta t\right), \tag{2}$$

where

Δt is the integration step,

T is the smallest oscillatory period with worthwhile contribution to the response [36],

Δt_{cr} is the upper bound on the integration step because of the linear theory of numerical stability [2, 23, 33],

Δt_{CFL} is the upper bound on the integration step in wave propagation problems and associated with spatial discretisation [37],

${}_f\Delta t$ is the step at which the excitation is digitised [1, 33, 35, 38] (and disappears when the excitation is continuous),

χ is defined as follows [1, 34, 35]:

$$\chi = \begin{cases} 10 & \text{when the behaviour is linear,} \\ 100 & \begin{array}{l}\text{when the behaviour is non-linear} \\ \text{and there is no impact,}\end{array} \\ 1000 & \begin{array}{l}\text{when the behaviour is non-linear} \\ \text{and there are impacts.}\end{array} \end{cases} \tag{3}$$

In many analyses, ${}_f\Delta t$ is the governing term in Equation (2), leading to $\Delta t = {}_f\Delta t$. Focusing on this special case, which will be even more popular in future (in view of the improvements in recording instrumentation), there are several major methods, that can accelerate the analyses by modifying the $\mathbf{f}(t)$; see [30–32, 39–42]. In view of the main features of one of these methods, i.e. the SEB THAAT (Step-Enlargement-Based Time History Analysis Acceleration Technique) [38, 39, 43–46], as listed below:

- Significant reduction in the analysis run-time; see Table 1,
- sufficiently accurate response history when the parameters are set properly; see e.g. [45, 46],
- simple formulation [1, 38, 39],
- good versatility; see Table 1,
- contribution of all the data of the original excitation in the new excitation [1, 38, 39],
- having a mathematical basis [39],
- having a formulation that depends on the excitation and not directly on the structural system [39],
- considerable number of the previous successful tests; see the review presented in Table 1,
- reducing the analysis run-time without increasing the use of in-core memory [1, 38, 39],

the SEB THAAT has a good potential to accelerate analyses of systems subjected to digitised excitations. In a review on the other methods [30–32, 40–42], three do not use the original excitation's total data in defining the new excitation [30–32], three take into account features of earthquakes and may be inappropriate for general structural dynamic problems [30, 31, 40], one produces new excitations digitised in unequal steps [40], the formulation and implementation of one is complicated [41], and for the method proposed in [42], the implementation is more complicated than the SEB THAAT [38, 39] and the method is tested on a small number of examples; see also Section 6. The SEB THAAT is therefore a good candidate for

Structural system	Reduction in the run-time [%]*	Details
Residential buildings	50–90	About 200 buildings with linear/non-linear behavior, regularity/irregularity in plan/height, under different earthquakes
A thirty-story building	50	A thirty-story steel three-dimensional frame subjected to two different excitations
Buildings subjected to multi-component earthquakes	75	An eight-story steel three-dimensional frame subjected to the three-component Abbar earthquake
Water tanks	67	Some experimental- and some real-sized water tanks subjected to different earthquakes
Bridges	30–80	About 20 real bridges with linear/non-linear behaviors, some with pre-stressed elements, subjected to different excitations including the multi-support
Power station, Cooling tower, Space structure, Silo	50–90	A couple of each structure, considering linear/non-linear behavior, different near-/far-field excitations and different integration schemes
Earth dams	<80	Several earth dams each subjected to several earthquake records
Milad telecommunication tower	50–70	Considering linear/non-linear behavior, near-/far-field excitations, and different integration schemes
Structural systems damped non-classically	30–90	Considering linear and non-linear behavior, and different integration schemes
Typical Iranian multi-span reinforced concrete bridges	>50	Developing IDA curves and fragility functions, considering deficient cap beam-column joint

* In the price of negligible inaccuracy.

TABLE 1. The main tests on the SEB THAAT's performance (see [1, 47, 48]).

further study to speed up the analysis of structural systems subjected to digitised excitations.

The objective of this paper is to review the SEB THAAT and propose a procedure for its clear application; such a procedure does not exist at present; see [38]. In continuation, after reviewing the SEB THAAT, its major challenge (i.e. proper selection of the SEB THAAT's parameter and clear application of the SEB THHAT [38]) is discussed. Then, to overcome the challenge, a procedure is proposed and its performance is evaluated via several examples, including a realistic example. A number of mainly practical issues are discussed later, and finally, the paper concludes with an overview of the achievements and an outlook for the future.

2. A review of the SEB THAAT

2.1. The focal idea, basics, and main formulation

Convergence is the most basic requirement of approximate computations [49, 50]. The analysis of Equation (1) by direct time integration is an approximate computation [1, 2, 23]. Therefore, convergence must be established for time integration calculations. Convergence of the responses produced by time integration implies that, for sufficiently small integration steps, by using smaller steps, the difference between the computed and exact responses should asymptotically vanish; see Figure 2 and [2, 23, 51]. In Figure 2, E is the error in arbitrary norm (the above-mentioned difference) [52] and q is the rate of convergence, generally equal to the integration method's order of accuracy [23, 39]. The analysis run-time is another important feature of direct time integration [1, 2, 23, 53].

When the right-hand-side of the first relation in Equation (1) is in digitised format, replacement of the excitation with an excitation digitised in larger steps can reduce the analysis run-time. The replacement is, however, an approximate computation, and the computed responses should continue converging to the exact responses. This idea, in addition to using all the data of the excitation in producing the new excitation, is formulated as the SEB THAAT, based on two mathematical facts, a broadly accepted convention and a realistic assumption [39]. The two facts are:

(1.) Consider Equation (1), its analysis by an integration method of order q, and an approximation of $\mathbf{f}(t)$, i.e. $\mathbf{f}_{\text{new}}(t)$, converging to $\mathbf{f}(t)$ with order q'. If $q' \geq q$, the analysis of Equation (1) by the integration method, after replacing $\mathbf{f}(t)$ with $\mathbf{f}_{\text{new}}(t)$, leads to responses that converge to the responses of the original Equation (1), with order q [39, 54].

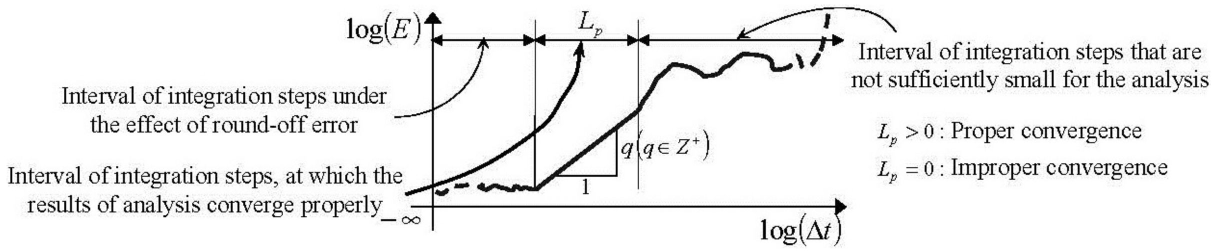


Figure 2. Typical trend of convergence in direct time integration analysis.

(2.) For a continuous function of a continuously changing variable x, i.e. $H(x)$, if Δx is a sufficiently small change of x, (O implies the big Oh operator) [55]:

$$H(x+\Delta x) + H(x-\Delta x) = 2H(x) + \mathrm{O}(\Delta x^2). \quad (4)$$

The convention is the second order of accuracy of majority of integration methods [2, 23]; and the realistic assumption is that, despite being available in digitised format, the $\mathbf{f}(t)$ in Equation (1) is a smooth function [55] of t. If these are valid, the SEB THAAT replaces $\mathbf{f}(t)$ (digitised in step ${}_f\Delta t$) with a new excitation $\mathbf{f}_{\text{new}}(t)$, digitised in step $({}_f\Delta t)_{\text{new}}$:

$$({}_f\Delta t)_{\text{new}} = n_f \Delta t, \qquad n > 1 \quad (5)$$

The new excitation, which preserves the response convergence and uses all the data of the original excitation, is defined as follows for integer values of n [1, 38, 39] (for real values of n, see [56]):

$$\mathbf{f}_{\text{new}}(t = t_i) = \begin{cases} \bar{\mathbf{f}}(t_i) & \text{when } t_i = 0 \\ a\bar{\mathbf{f}}(t_i) + (1-a)\displaystyle\sum_{k=1}^{n'} b_k \left[\bar{\mathbf{f}}\left(t_i + k\,{}_f\Delta t\right) + \bar{\mathbf{f}}\left(t_i - k\,{}_f\Delta t\right)\right] & \text{when } t_i = t_1, t_2, \dots, t'_{\text{end}} - n_f\Delta t \\ \bar{\mathbf{f}}(t_i) & \text{when } t_i = t'_{\text{end}} \end{cases} \quad (6)$$
$$\mathbf{f}_{\text{new}}(t \neq t_i) = 0$$

where

$$t_i = i(n\,{}_f\Delta t), \qquad i = 0, 1, 2, \dots, \frac{t'_{\text{end}}}{n\,{}_f\Delta t}, \quad (7)$$

$$\bar{\mathbf{f}}(t_i) = \begin{cases} \mathbf{f}(t_i) & \text{when } 0 \le t_i \le t_{\text{end}}, \\ \mathbf{O} & \text{when } t_{\text{end}} < t_i \le t'_{\text{end}}, \end{cases} \quad (8)$$

$$a = \frac{1}{2}, \qquad b_k = \frac{1}{2n'}, \quad (9)$$

$$n' = \begin{cases} n-1 & \text{when } t_i = t_1 = n\,{}_f\Delta t \text{ or } t_i = t'_{\text{end}} - n\,{}_f\Delta t, \\ j \quad \left(j \in Z^+, \ \dfrac{n}{2} - 1 < j \le \dfrac{n}{2}\right) & \text{when } n\,{}_f\Delta t < t_i < t'_{\text{end}} - n\,{}_f\Delta t, \end{cases} \quad (10)$$

and considering that $\frac{t_{\text{end}}}{{}_f\Delta t}$ is not necessarily a positive integer,

$$t'_{\text{end}} = \begin{cases} t_{\text{end}} & \text{when } \dfrac{t_{\text{end}}}{n\,{}_f\Delta t} \in Z^+, \\ l(n\,{}_f\Delta t) & \text{when } \dfrac{t_{\text{end}}}{n\,{}_f\Delta t} \notin Z^+, \quad t_{\text{end}} < l(n\,{}_f\Delta t) < t_{\text{end}} + n\,{}_f\Delta t. \end{cases} \quad (11)$$

Implementation of Equations (5)–(11) is reviewed in Figure 3. Application of the SEB THHAT to analysis of Equation (1), using a specific value of n, implies:

(1.) computation of $\mathbf{f}_{\text{new}}(t)$,

(2.) replacing $\mathbf{f}(t)$ with $\mathbf{f}_{\text{new}}(t)$ in Equation (1),

(3.) time integration of Equation (1), with the integration step, obtained from Equation (5).

In view of Equations (2) and (5)–(11) and Figure 3, for a clear and effective application of the SEB THAAT, the value of n must be set carefully. Assigning an excessively large value to n can lead to very inaccurate responses, while assigning too small a value to n will prevent the SEB THAAT from reducing the analysis run-time, in accordance with the potential of the problem, the analysis, and the SEB THAAT.

In view of Equation (5) and because of the small run-time needed to implement Equations (6)–(11) (compared to that of direct time integration), the SEB THAAT can reduce the analysis run-time. However, the response of the analysis after applying the SEB THAAT should differ negligibly from the response computed ordinarily. With attention to Equation (2), a comment to limit the inaccuracy is to satisfy (see [1, 38]):

$${}_f\Delta t < \min\left(\frac{T}{\chi}, \Delta t_{\text{cr}}, \Delta t_{\text{CFL}}\right). \quad (12)$$

2.2. The literature

Since its launch in 2008, research on the SEB THAAT has progressed in two main directions. In one direction, the performance of the SEB THAAT is under test (see Table 1), considering different structural systems, different non-linear behaviours, different damping mechanisms, different integration methods, different digitised excitations, etc., and starting with single degree of freedom systems [39], recently focusing on large systems such as the Mi-

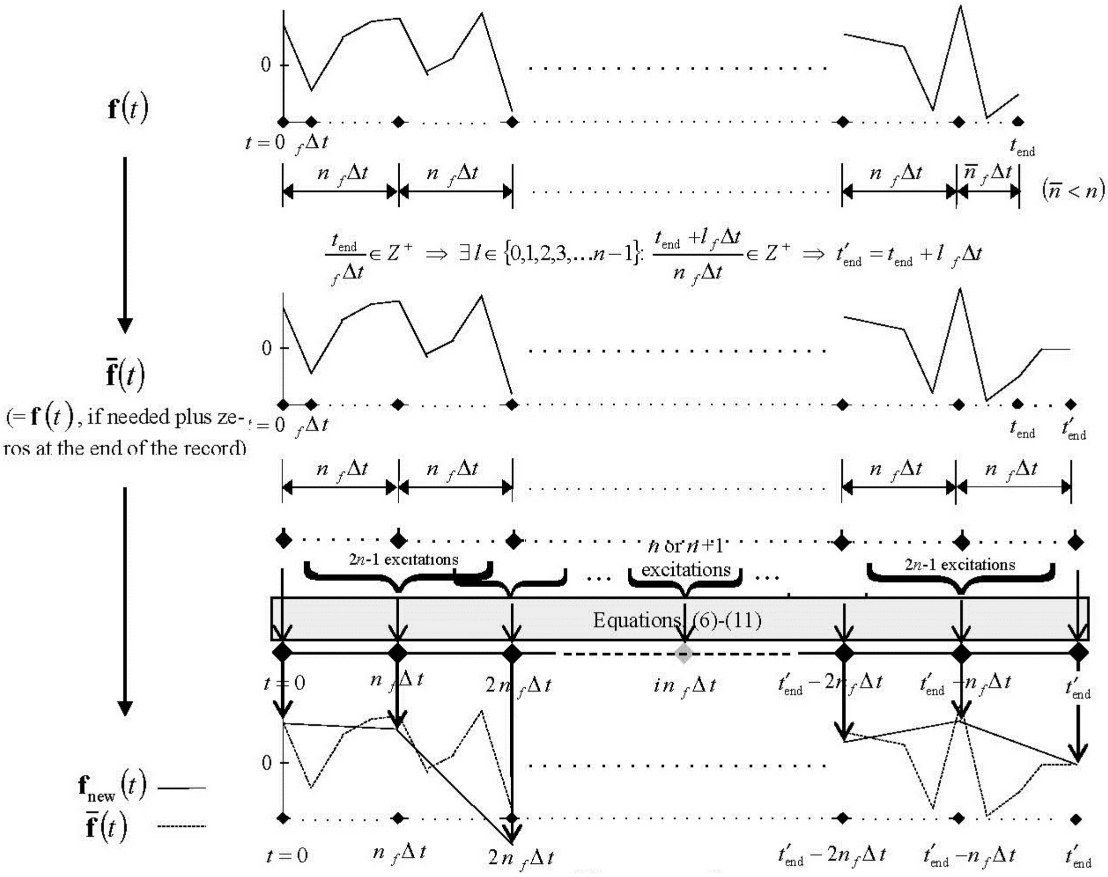


FIGURE 3. The process of replacing $\mathbf{f}(t)$ with $\mathbf{f}_{\text{new}}(t)$ using Equations (5)–(11).

lad telecommunication tower [57], earth dams and bridges [47, 48, 58, 59], and the Lower San Fernando Dam [60]. Each test looks for answers to the following three questions:

(1.) Is Equation (12) valid?

(2.) When Equation (12) is valid, is there any real value for n that, after using the SEB THAAT, leads to sufficiently accurate target responses? Specifically, is the following equation valid:

$$\text{Equation (12) holds} \Rightarrow \exists\, n > 1\colon \mathbf{R}_{\text{new}} \cong \mathbf{R}? \quad (13)$$

($\mathbf{R}$ and $\mathbf{R}_{\text{new}}$ are the target responses obtained from the ordinary analysis and the analysis using the SEB THAAT, respectively.)

(3.) When Equation (12) is valid, is Equation (13) valid for all values of n satisfying:

$$n \le n_{\max} = \frac{1}{{}_f\Delta t} \min\left(\frac{T}{\chi}, \Delta t_{\text{cr}}, \Delta t_{\text{CFL}}\right)? \quad (14)$$

(Equation (14) is derived by replacing the ${}_f\Delta t$ in Equation (12) with $n_{\max}\,{}_f\Delta t$.)

For the problems reviewed in Table 1, the answers to the three questions are positive, and the reduction in the run-times is considerable [1, 43, 45–48, 57–67]. Even more, for some tests, Equation (13) holds for values of n larger than the $n_{\max}$ in Equation (14), e.g. see [66]. Furthermore, for two tests [57, 61], the accuracies increase after applying the SEB THAAT, i.e. simultaneous reduction in the analysis run-time and the error of the target response. Besides, the results of the few tests carried out on systems with wave propagation behaviour are satisfactory [1, 47, 58, 60]. Also, in one test, the decrease in the run-time is greater, when the behaviour is non-linear [57]. The successful performance of the SEB THAAT, when the excitation record is related to near- or far-field earthquakes, is briefly demonstrated [60], as well. Impressive application of the SEB THAAT to the analysis of multistory steel structure buildings when the structural plan is regular or irregular in plan or height is another great achievement [44, 61, 66, 68]. It is also displayed that the application of the SEB THAAT in analyses essential for fragility study, though is together with verification computations, can considerably reduce the total run-time [47, 48]. Finally, some tests on the application of the SEB THAAT to analyses other than the solution of Equation (1) have been successfully carried out; see [38, 69–71].

The other main direction of the research seeks answers to various conceptual questions. First, it is shown that the excitation can be multi-component [72]. The effect of non-linearity on the performance and specifically the accuracy of the responses after application of the SEB THAAT are studied next. As the result, when the non-linearities are modelled properly

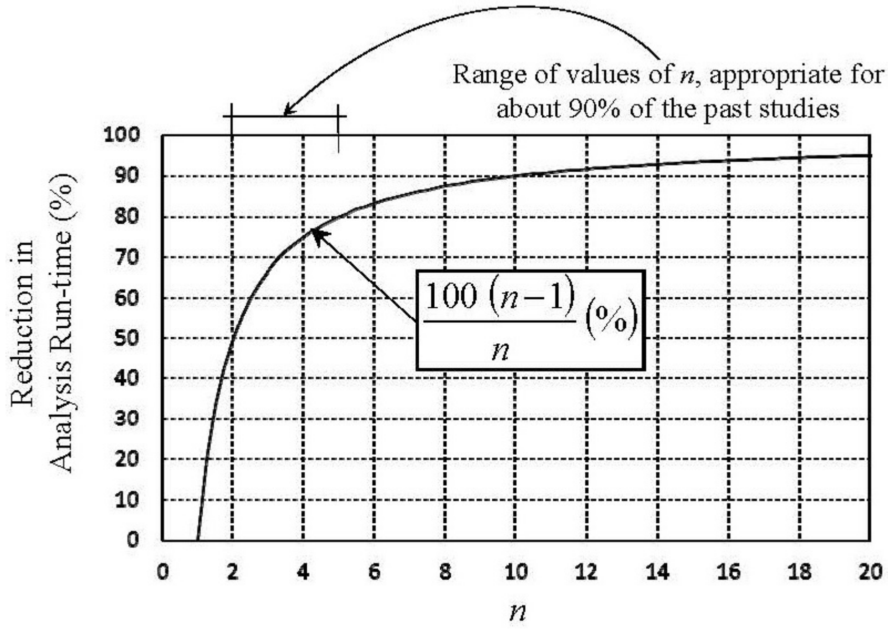


FIGURE 4. Reduction in analysis run-time as a function of the scale n, for linear direct time integration computations accelerated by the SEB THAAT.

and adequate values are assigned to the parameters of the non-linear solution, the SEB THAAT's performance can be conceptually similar in linear and non-linear analyses [46]. Meanwhile, the T in Equation (2) should be related to the target response [73]. Then, the practical preference for considering an upper bound for the enlargement scale, n, is mentioned (see [1, 74]). The SEB THAAT is later compared with direct down sampling; and as a main observation, though the down sampling can lead to a good accuracy and analysis run-time reduction in some tests, the performance of the SEB THAAT is never weaker [64]. In the same study [64], it is demonstrated, that for the SEB THAAT to be successful, Equation (1) should not necessarily be the result of finite element method [4, 75]; finite volume method [76] is also acceptable. It is then studied whether the inaccuracy because of the SEB THAAT can cause numerical instability [77]. As the outcome, when Equation (2) holds, the responses obtained from linear analyses are stable, regardless of whether the SEB THAAT is applied. Meanwhile, even when the integration method's order of accuracy is different from two, the SEB THAAT can be successful [78]. The effects of the SEB THAAT on the run-times of linear and non-linear time integration analyses are compared as well [79]; in addition, it is also shown that, in contrast to non-linear analyses, in linear analyses, the reductions can be determined, in terms of the enlargement scale (see [1]). The next study [80] was on values of a and b_k different from those introduced in Equation (9) and subjected to the following convergence-based restriction (inherited from [39]):

$$\sum_{k=1}^{n'} b_k = 0.5. \tag{15}$$

As the result, though Equation (9) is not always the best selection, it is the best selection when considering different cases of the parameters in Equation (1) and different integration methods. The other question was how to extend Equations (6)–(11) to an arbitrary real value of n larger than one; an appropriate way is presented in [56]. Some initial studies are also performed on the frequency content of the inaccuracies due to the SEB THAAT (see [1]). Recently, the performance of the SEB THAAT has been studied when the structural system is non-classically damped [43]. As the result, for majority of time integration methods, the performance is independent of the type of viscous damping. In a very recent study, it has been demonstrated that for steel structural buildings with 5–20 floors, the SEB THAAT can be reliably used, considering $n = 2$ [44]. Finally, the performance of the SEB THAAT compared to some other analysis acceleration techniques is discussed in [42].

2.3. A MAJOR CHALLENGE

As discussed in [38, 44, 60], a major challenge for the SEB THAAT is the clear and practical application of the SEB THAAT. For a better explanation, with reference to Figure 4 and Equation (12), the reduction in the analysis run-time and the accuracy of the target response can be very sensitive to the problem under investigation. As a direct consequence, for a clear practical application of the SEB THAAT, assigning appropriate values to the SEB THAAT's parameter, i.e. n, is an important challenge.

In addition, time history analysis of structural systems, while irreplaceable in many cases [81], is generally time consuming [1, 23, 39, 82], especially when the analysis is a part of a probabilistic or optimisation computation, the structural system is very large, or the structural behaviour is highly non-linear or very oscillatory; see [82–85]. In application of the SEB THAAT, the reductions in the analysis run-time can be significant (see Table 1), even compared to other analysis acceleration methods (see [1] and Section 6). Consequently, it is reasonable to use the SEB THAAT to reduce the large computational efforts in many real time history analyses. Developing a practical way for clear and simple application of the SEB THAAT is therefore a necessity, for which, the SEB THAAT's enlargement scale, n, should be set carefully.

Moreover, taking into account Equations (5)–(11) and (13), n is the only parameter to be set for application of the SEB THAAT (in addition to the parameters of the ordinary time history analysis). Accordingly, the main challenge for a clear practical application of the SEB THAAT is determining the appropriate value of n. Some ambiguities are stated next.

First, currently the only relation for determining the n is the following inequality:

$$1 < n \leq n_{\max} = \frac{1}{{}_f\Delta t} \min\left(\frac{T}{\chi}, \Delta t_{\mathrm{cr}}, \Delta t_{\mathrm{CFL}}\right). \quad (16)$$

Secondly, given its definition, T in Equation (16) cannot be easily determined or estimated, especially prior to the analysis. Next, that the definition of T, given immediately after Equation (2), and in particular the notion of "worthwhile" therein, is not clear. Then, the $\frac{T}{\chi}$ in Equation (16) is not a precise criterion for accuracy [1, 86–90]. Finally, the $\frac{T}{\chi}$ in Equations (2) or (16) is the only term in these equations that relates the integration step to the structural system and behaviour. Consequently, a direct determination of n is complicated and can be costly, contradicting the purpose of the SEB THAAT, i.e. reducing the analysis run-time. An alternative can be to approximate the appropriate value of n by some upper estimation of n and attempt to correct the estimation in several steps. Meanwhile, for special groups of analyses, simple reliable values can be assigned to n; for a recent achievement, see [44].

3. Simple application of the SEB THAAT

3.1. Theoretical bases

The current approach to applying the SEB THAAT is to simply assign a value to n, based on the experience. In this section, the current approach is replaced with an algorithm that, starting from an upper estimation of n, after a number of repetitive time integration computations, assigns a value to n, appropriate for the accuracy of the target response. The new approach is consistent with the comments, in:

(1.) structural dynamics [33],

(2.) numerical solution of ordinary initial value problems [91],

(3.) the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92].

Accordingly, it is reasonable to terminate the iterations of the new approach using the criterion in the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92], that is, after repeating a time integration computation with half steps, the absolute relative difference of the two peak target responses should not exceed 5 %. It is reasonable to start from $n = 20$; see Figure 4 and [43]. Therefore, what remains unclear is mainly the details of each successive analysis, with respect to the previous analyses. Specially, it should be determined how to change the n and the Δt. For non-linear analyses, additional details, e.g. the non-linear tolerance, need special attention, as well.

Using a subscript on the right to introduce the sequence of time integration computation and paying attention to the discussion above and Equation (5) leads to:

$$\Delta t_1 = n_1\, {}_f\Delta t, \quad n_1 = 20. \quad (17)$$

With careful attention to Equation (17), the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92], and the comments in structural dynamics and numerical analysis of initial value problems [33, 91], not only the n but also the ${}_f\Delta t$ can change from one time integration computation to the next time integration computation. Even in ordinary analyses, when repeating a time integration computation with half an integration step, the ${}_f\Delta t$ is halved by linear interpolation of the excitation data [33, 35, 92, 93]. Consequently, Equation (17) would rather be replaced with:

$$\Delta t_1 = n_1\, {}_f\Delta t_1, \quad n_1 = 20, \quad {}_f\Delta t_1 = {}_f\Delta t. \quad (18)$$

It is also worth noting that the successive computations starting from Equation (17) cannot consider integration steps smaller than ${}_f\Delta t$ (see the inequality in Equation (5)), whereas there is no limitation when starting from Equation (18).

In view of the details of the SEB THAAT, the $\mathbf{f}_{\mathrm{new}}(t)$ corresponding to $j = 1$ can be obtained by first replacing the digitised $\mathbf{f}(t)$, with a record, $\mathbf{g}(t)$, digitised in steps equal to ${}_f\Delta t_1$, using linear interpolation; and then using Equations (6)–(11) and $n = n_1$, to replace $\mathbf{g}(t)$ with $\mathbf{f}_{\mathrm{new}}(t)$. This approach can be used for all the successive computations, i.e. arbitrary value of j (for the first computation, the linear interpolation is trivial). Accordingly, by extending Equation (18) to:

$$\Delta t_j = n_j\, {}_f\Delta t_j, \quad (19)$$

determination of how n_j and ${}_f\Delta t_j$ should change with j is to be clarified.

Comparing with the successive time integration computations in ordinary time history analysis, where

j	1	2	3	4	5	6	7	8	...
n_j	20	10.5	5.75	3.375	2.1875	1.59375	1.296875	1.0742175	$\cdots > 1$
$\frac{\Delta t_j}{\Delta t_{j-1}}$	-	0.2625	0.2738	0.2935	0.3241	0.3643	0.4069	0.4428	$\cdots < 0.5$
$\frac{\Delta t_j}{{}_f\Delta t}$	20	5.25	1.4375	0.421875	0.13671875	...	...	...	$\cdots > 0$

TABLE 2. A numerical review of Equation (22).

$\Delta t_j = {}_f\Delta t_j$ (see [33, 35, 92]), ${}_f\Delta t_j$ can be considered representing the inaccuracy because of the integration method's approximation [1, 2, 23]. Similarly, with regard to Equation (5), n_j represents the inaccuracy due to the SEB THAAAT. Therefore, with attention to the theoretical bases of the SEB THAAT [38, 39], it is reasonable to preserve the consistency between the changes of the two sources of inaccuracy. Given this, the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92], and the fact that, at $n = 1$ and ${}_f\Delta t \to 0$, the inaccuracies due to the SEB THAAT and the approximations of the integration method disappear, it is reasonable to preserve the consistency by:

$$\frac{n_j - 1}{n_{j-1} - 1} = \frac{{}_f\Delta t_j}{{}_f\Delta t_{j-1}} = \frac{1}{2}, \qquad j = 2, 3, 4, \ldots \tag{20}$$

Specifically, the reason for the "$\frac{1}{2}$" in Equation (20) is the tradition of halving the integration steps to check the accuracy of the target response in engineering and science; see [33, 35, 91, 93]. Together with Equations (18) and (19), Equation (20) leads to the following integration steps:

$$\begin{aligned} &\Delta t_j = n_j\, {}_f\Delta t_j, \qquad j = 1, 2, 3, \ldots \\ &j = 1: \quad n_1 = 20, \qquad {}_f\Delta t_1 = {}_f\Delta t \\ &j = 2, 3, 4, \ldots: \quad n_j = \frac{n_{j-1} + 1}{2}, \quad {}_f\Delta t_j = \frac{\Delta tj - 1}{2} \end{aligned} \tag{21}$$

and the fact that (see also Table 2):

$$\begin{aligned} 0.25 < \frac{\Delta t_j}{\Delta t_{j-1}} &= \frac{n_j}{n_{j-1}} \frac{{}_f\Delta t_j}{{}_f\Delta t_{j-1}} \\ &= \frac{\frac{n_{j-1}+1}{2}}{n_{j-1}} \frac{\frac{{}_f\Delta t_{j-1}}{2}}{{}_f\Delta t_{j-1}} \\ &= \frac{1}{4} \frac{n_{j-1}+1}{n_{j-1}} < 0.5, \qquad j = 2, 3, 4, \ldots \end{aligned} \tag{22}$$

From Equation (22) and Table 2, the following points can be concluded:

(1.) The ending criterion of the time history analysis in the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92] cannot be used, when setting the integration step according to Equation (21). The reason is that, different from what is implied in Equation (21) (see the third row in Table 2), in the procedure in the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92], the integration step halves with each new time integration. Using P_j to introduce the peak target response in the j^{th} computation, the correct ending criterion is (see Appendix A, for a proof attempt):

$$\frac{|P_{j-1} - P_j|}{P_j} \le \frac{1}{60} \left[\left(\frac{4n_{j-1}}{n_{j-1} + 1} \right)^2 - 1 \right], \qquad j = 2, 3, 4, \ldots \tag{23}$$

Note that, using $n_{j-1} = 1$ in Equation (23) simplifies Equation (23) to the ending criterion in the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92].

(2.) A procedure that uses Equation (21) for computing Δt_j can be continued endlessly, until the ending criterion is satisfied, i.e. there is no lower bound on Δt_j. The reason is implied in the relation leading to Δt_j; see Equation (21) and Table 2.

(3.) As it should, the integration step Δt_j converges to zero (see the last row of Table 2).

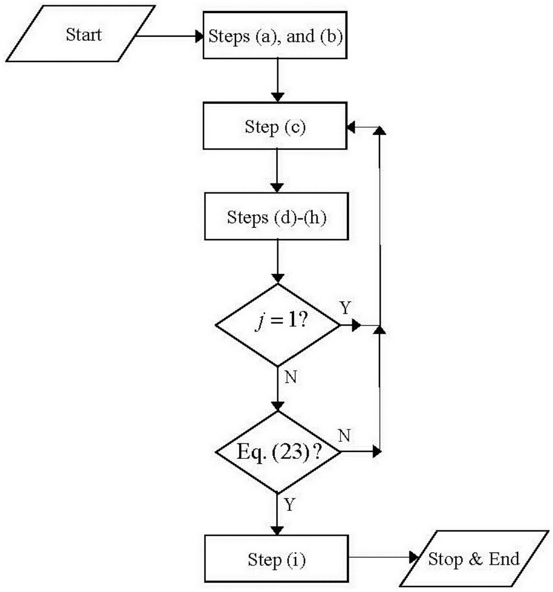


FIGURE 5. A procedure for simple and clear application of the SEB THAAT.

A procedure to simply and clearly apply the SEB THAAT is presented next.

3.2. The procedure

By using the following procedure, assigning values to n, the main parameter of the SEB THAAT, can be automated, eliminating concerns about determination of n (see Figure 5):

(a) Select the target response, the integration method (preferably unconditionally stable; see [1, 2, 23, 94]),

j	1	2	3	4	5	6
$\bar{\delta}_j$	1.E−2	1.E−6	1.E−9	1.E−11	1.E−13	1.E−14

Table 3. Values of non-linear tolerance in application of the proposed procedure.

and for non-linear problems, set the non-linear solution details.

(b) $j = 0$.

(c) $j = j + 1$.

(d) Compute n_j, using:

$$n_1 = 20, \tag{24a}$$

$$n_j = \frac{n_{j-1}+1}{2}, \qquad j = 2, 3, 4, \ldots \tag{24b}$$

(e) Use linear interpolation to change the $\mathbf{f}(t)$ at the right hand side of Equation (1) to $\mathbf{g}(t)$, digitised in steps equal to ${}_f\Delta t_j$, defined as follows (when $j = 1$, $\mathbf{g}(t) = \mathbf{f}(t)$):

$${}_f\Delta t_j = \frac{{}_f\Delta t}{2^{j-1}}. \tag{25}$$

(f) Use Equations (6)–(11) to change $\mathbf{g}(t)$ to $\mathbf{f}_{\text{new}}(t)$, digitised in steps equal to Δt_j:

$$\Delta t_j = n_j\, {}_f\Delta t_j. \tag{26}$$

(g) Time integrate Equation (1), considering $\mathbf{f}_{\text{new}}(t)$ as $\mathbf{f}(t)$, and Δt_j as the integration step; for non-linear problems, use the non-linear tolerances, $\bar{\delta}_j$, stated in Table 3 (see the comments in [95] and the difference between the integration steps in the last row of Table 2), and do not stop the computation when the iteration of non-linear solution fails (see [96]).

(h) Compute the peak target response, as P_j.

(?) If only one-time integration computation is carried out, return to Step (c).

(?) If the last two peak target responses computed in Step (h) do not satisfy Equation (23), return to Step (c).

(i) Accept the last time integration computation and response as final.

(j) Stop.

Obviously, for application of the above procedure, no parameter regarding the SEB THAAT needs to be set in advance, a measure of inaccuracy is computed (see Appendix A), and there is no limitation for the application. These are remarkable achievements in terms of simplicity, availability of a measure of inaccuracy, and versatility. Nevertheless, it is also essential to study the accuracy and computational effort of applying the SEB THAAT according to the proposed procedure and compare the results with those summarised in Table 1.

3.3. Complementary points

The procedure proposed in the previous section, besides eliminating the need to select a value for n, has removed the T, χ, Δt_{cr}, and Δt_{CFL} from the analysis. This is an additional significant achievement, simplifying the analysis even more. The removal of these four parameters can be explained by considering the role of these four parameters in time history analysis. In short, these parameters are only necessary to maintain the accuracy (including numerical stability) of the target response [1, 2, 23, 33, 34]. Meanwhile, the accuracy is checked in the last decision-making step of the proposed procedure (just before the Step (i)). Therefore, it is reasonable to consider the accuracy control according to Equation (2) redundant and discard it.

A main difference between the previous applications of the SEB THAAAT and applying the SEB THAAT according to the proposed procedure is that while in the previous applications there was only one time integration computation, several time integration computations are essential when using the proposed procedure. In addition, a measure of the accuracy is available in terms of the peak target response when using the proposed procedure. Accordingly, when comparing the proposed procedure with the ordinary analysis, it is reasonable to carry out the ordinary analysis sequentially, each time with half steps, and end the analysis iterations with the 5 % criterion of NZS 1170.5:2004, as well. (This is in agreement with the existing comments; see [33, 91, 93].) Considering this and Figure 5, we can expect the proposed procedure to be neither complicated nor computationally expensive.

Finally, it is worth noting that adding the "preferably unconditionally stable" to the Step (a) of the proposed procedure implies that using an unconditionally stable method for the analysis is preferable, but not obligatory [2, 23, 94]. (Only, unconditionally unstable methods should not be used.) Several explanations can be presented for this statement in Step (a) of the procedure. Firstly, the procedure involves repeating time integration computations, and hence even when the response is inaccurate in one-time integration computation, it may be sufficiently accurate in the subsequent computations. The run-time needed for the inaccurate computation is negligible compared to the total analysis run-time as well; see the last row of Table 2 and [1]. The second explanation is that the time history analysis and time integration are mostly for non-linear analyses [1, 2, 23], for which satisfying the requirements of linear stability may be insufficient; see [1, 86–90, 97]. And as the final explanation, using

j	1	2	3	4	5	6	$\geq$7
$\bar{\delta}_j$	1.E−2	1.E−4	1.E−6	1.E−8	1.E−10	1.E−12	1.E−14

Table 4. Values of non-linear tolerance for ordinary sequential time integration computations [95].

Equations (24)–(26), similarly for all problems, regardless of the integration method, makes the analysis to be simpler, more attractive, and perhaps even more efficient for large problems.

4. Illustrative examples

4.1. Preliminary notes

Ground motions are generally available in a digitised format [33]. For this reason, and because of the role of the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92] in the presented discussions, the structural systems in this section are considered to be subject to ground acceleration. Accordingly, in Equation (1):

$$\mathbf{f}(t) = -\mathbf{M}\boldsymbol{\Gamma}\ddot{u}_g(t), \tag{27}$$

where

$\ddot{u}_g(t)$ implies the ground acceleration,

$\boldsymbol{\Gamma}$ is a vector with the size of the degrees of freedom, needed for matrix multiplication and considering spatial changes of $\ddot{u}_g$ [33].

In addition, similarly to the majority of the past studies on the SEB THAAT (see Table 1), the examples here have a structural dynamic nature, where the members of $\boldsymbol{\Gamma}$ are all equal to one. Furthermore, as explained in Subsection 3.3, for comparing the ordinary analysis with the analysis according to the proposed procedure, the former is consisted of sequential time integration computations. The procedure is similar to the proposed procedure, with roots in conventional analyses (see [33, 35, 92]; only slight changes (including replacement of Table 3 with Table 4 and using $n_j = 1$) are implemented in the proposed procedure. For a better explanation, for ordinary analyses, first, a time integration computation is carried out with ${}_f\Delta t$ as the integration step and the non-linear tolerance recommended in [95]; see Table 4. The computation is then repeated, with updated parameters, including half integration steps, new non-linear tolerances (see Table 4), and new excitations obtained using linear interpolation. If the absolute relative difference of the two peak target responses is not more than 5 %, the last response is final. Otherwise, the computation is repeated, until convergence of the peak target response is reached. Meanwhile, as demonstrated in [96], similar to the analyses using the proposed procedure for application of the SEB THAAT, the time integrations do not stop when the non-linear solutions fail.

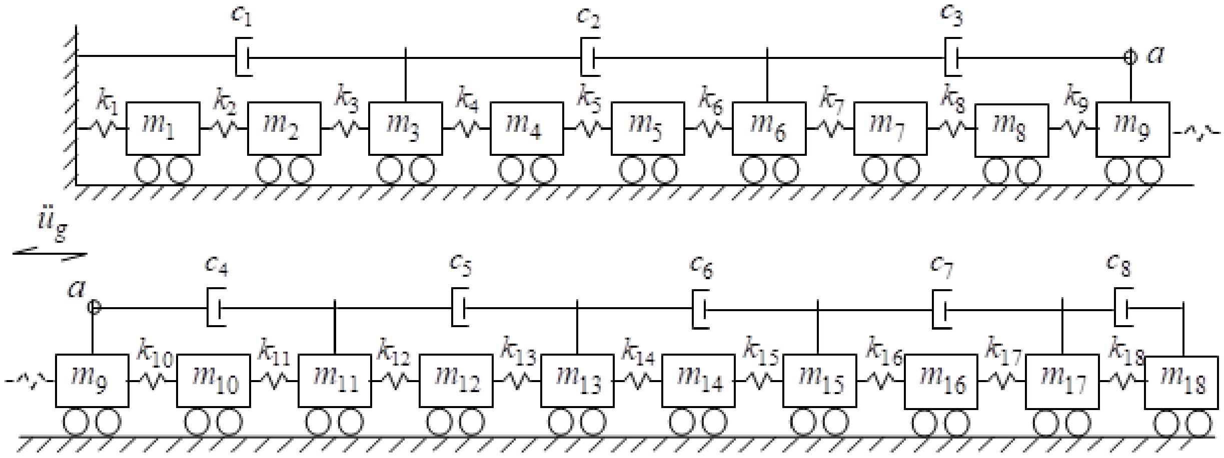
A question that may arise here is why in the first time integration computation of the ordinary time history analysis, the selection of the integration step is not according to Equation (2) or its slightly modified version in the New Zealand Seismic Code [35, 92]. A main reason is that, given the objective of the paper and the ending criterion of the proposed procedure, it is sufficient to show that the proposed procedure is clear in application and can notably reduce the analysis run-time (compared to the ordinary analyses), for many cases. Considering these, for the sake of simplicity and consistency with the analyses according to the proposed procedure, it is reasonable to consider one of the terms in the right hand side of Equation (2), and use ${}_f\Delta t$ as the integration step of the first time integration computation of the ordinary time history analysis. For majority of cases, this approach (using an integration step in the first analysis larger than the result of Equation (2)) will reduce the analysis run-time of ordinary time history analysis. The reduction in analysis run-time due to the use of the proposed procedure will then be a lower-bound of the true reduction in the analysis run-time. When needed, Equation (2) can be considered for determination of the integration step in the first time integration computation of the ordinary time history analysis.

Accuracy of the responses after applying the SEB THAAT according to the proposed procedure is determined by comparing the responses with the responses obtained from the ordinary analysis. The run-times essential for the analyses are compared in view of the total numbers of integration steps. Accordingly, in this section, fractional time stepping, with the maximum number of non-linear iterations equal to five (as conventional) [98, 99], is used for the non-linear solution. Other choices for measuring the analysis run-time and non-linear solution are used in the study of a realistic example in Section 5. All values are given in the International System of Units (SI).

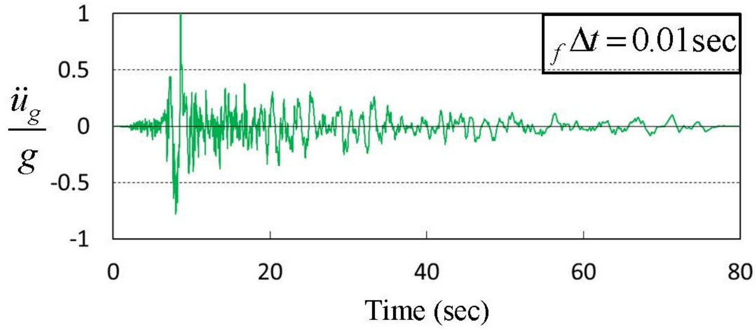
4.2. Example one: A simple non-linear problem

Figure 6 shows a preliminary model of a tall building's structural system, subjected to a ground acceleration. g stands for the acceleration of gravity, equal to $9.81\,\mathrm{m\,s^{-2}}$, and Table 5 reviews the model's main properties. Specifically, the stiffness is linear-elastic/perfect-plastic considering unloading (u_{y_i} is the yield displacement of the i^{th} spring), and hence the behaviour may be non-linear.

In Step (a) of the proposed procedure, the base shear is set as the target response, and the C-H method [100] ($\rho_\infty = 0.7$) is set for time integration. Steps (b)–(d) lead to $n_1 = 20$. As the result of Steps (e) and (f), Figure 7a shows the excitation in the first time integration computation (“new” as



(A). Structure's model.



(B). Ground acceleration.

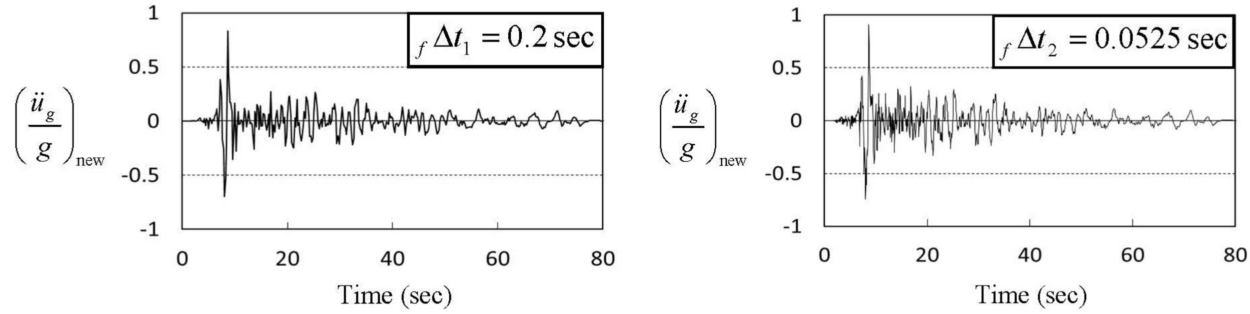
FIGURE 6. The structural system in the first example.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$10^{-9} \times m_i$	3	3	3	3	3	3	3	3	1.5	1.5	1.5	1.5	1.5	1.5	0.5	0.5	0.5	0.5
$10^{-9} \times k_i$	2	2	2	2	1.2	1.2	1.2	1.2	0.6	0.6	0.6	0.6	0.6	0.6	0.1	0.1	0.1	0.1
$10^{-9} \times c_i$	1.2	0.8	0.6	0.25	0.25	0.15	0.05	0.02	-	-	-	-	-	-	-	-	-	-
u_{y_i}	0.8	0.8	0.8	0.8	0.8	0.8	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.3	0.3	0.3	0.3

TABLE 5. Main properties of the structural system in the first example.

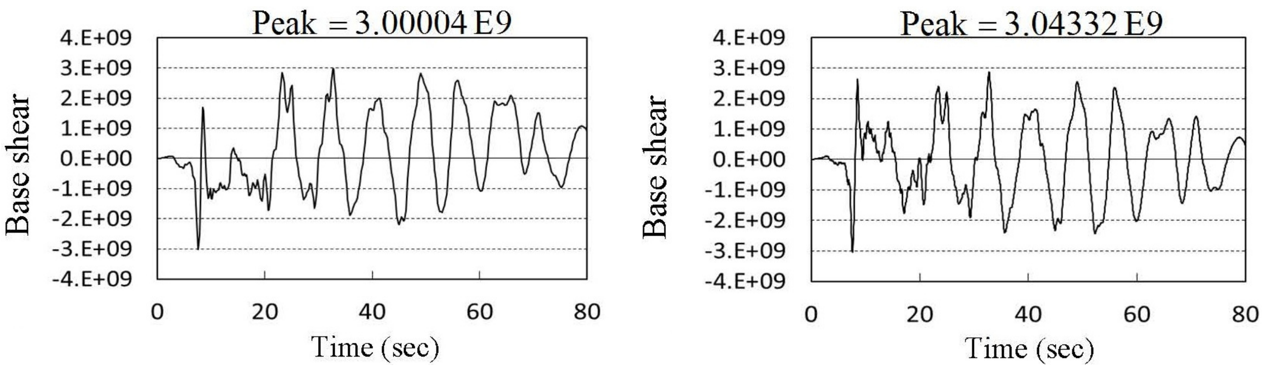
a subscript implies that the argument is related to the SEB THAAT's application using the proposed procedure). In Step (g), the non-linear tolerance is set to 10^{-2} that, when used together with the integration step set in Step (f), results in the time history reported in Figure 8a for the target response, the peak of which is stated at the top of the figure. In view of the number of analyses carried out, the computation proceeds from Step (h) to Step (c). Steps (c) and (d) lead to $n_2 = 10.5$, using which, Steps (e) and (f) result in Figure 7b as the excitation record. In Steps (g) and (h), using the tolerance in Table 3 corresponding to $j = 2$ and the integration step equal to the step of the excitation in Figure 7b, leads to a time integration computation with the results shown in Figure 8b. According to the number of time integration computations performed, it is then checked whether the peak target responses reported in Figures 8a and 8b satisfy Equation (23). The answer is positive, and Step (i) introduces Figure 8b as the final response; and the procedure stops in Step (j).

In the ordinary analysis, a time integration computation is first performed with steps equal to $_f\Delta t$ (=0.01 s) and a non-linear tolerance equal to 10^{-2} (see Table 4). The resulting time history of the target response and the peak value are shown in Figure 9a. The second computation is carried out with half steps and a non-linear tolerance equal to 10^{-4} (see Table 4). The consequence is shown in Figure 9b. Given the peak values noted at the top of Figures 9a and 9b, the difference in the peak target responses is much less than 5 % and hence the response displayed in Figure 9b is final. The run-times of the two analysis approaches are compared in Table 6 (where the red oval shapes refer to the run-time details of the final computations and the red numbers are used to compare the run-times), and the good accuracy is evident when comparing Figures 8b and 9b. The result is an 87.26 % reduction in the analysis run-time, at the cost of a visually unrecognisable change in the accuracy of the target response. Therefore, by using the proposed procedure, the SEB THAAT may be



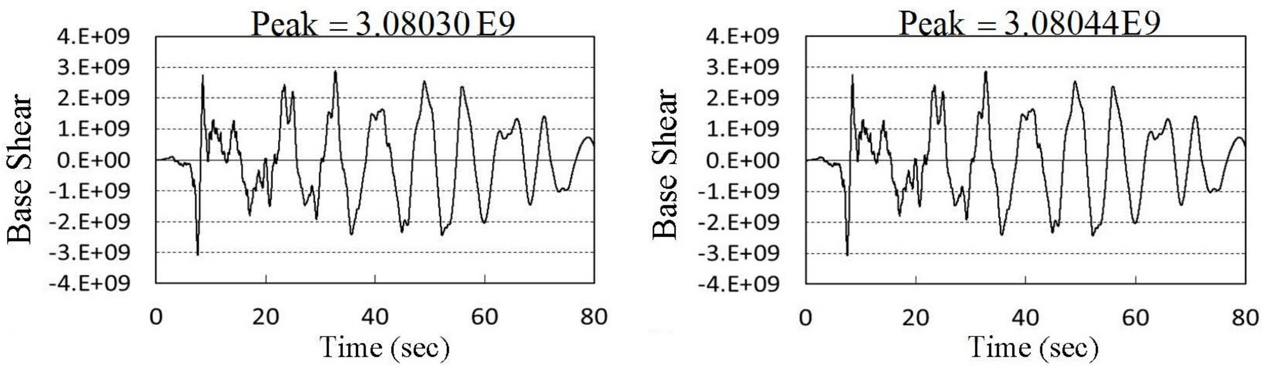
(a). $j = 1$. (b). $j = 2$.

Figure 7. Excitation records obtained from Steps (e) and (f) of the proposed procedure for the first example.



(a). $j = 1$. (b). $j = 2$.

Figure 8. History of the target response and the peak value as the result of Steps (g) and (h) of the proposed procedure for the first example.



(a). $j = 1$. (b). $j = 2$.

Figure 9. History of the target response and the peak value obtained from ordinary time history analysis of the first example.

easily applicable (without worrying about the value of n) and significantly speed up non-linear time history analyses. Finally, it should be noted that the behaviour of the structural system is indeed non-linear, albeit slightly, as shown in Figure 10.

4.3. Example two: An interesting linear problem

The model in Figure 11a is subjected to the ground acceleration in Figure 11b. The target response, S, is the sum of the kinetic energy E_K and the potential energy E_P (equal to the input energy of the earthquake minus the energy damped in the structure), i.e.,

$$S = E_K + E_P. \tag{28}$$

Given Equations (1) and (28), by denoting the displacements of the three masses by u_1, u_2, and u_3, and the corresponding velocities by $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$, the target response S can be expressed as [93]:

$$\begin{aligned} S =& 5(\dot{u}_1^2 + \dot{u}_3^2) + 10\dot{u}_2^2 \\ &+ 10{u_1}^2 + 5(u_3 - u_1)^2 + 20{u_3}^2. \end{aligned} \tag{29}$$

Starting the study with the proposed procedure, in Step (a), S is the target response and the integration

Analysis iteration	Ordinary sequential analyses			Proposed procedure		
	Integration step [s]	Number of integration steps	Total number of integration steps	Integration step [s]	Number of integration steps	Total number of integration steps
1	0.01	8 006	8 006	0.2	606	606
2	0.005	16 190	24 196	0.0525	2477	3 083

Table 6. Study of the analysis run-times of the first example.

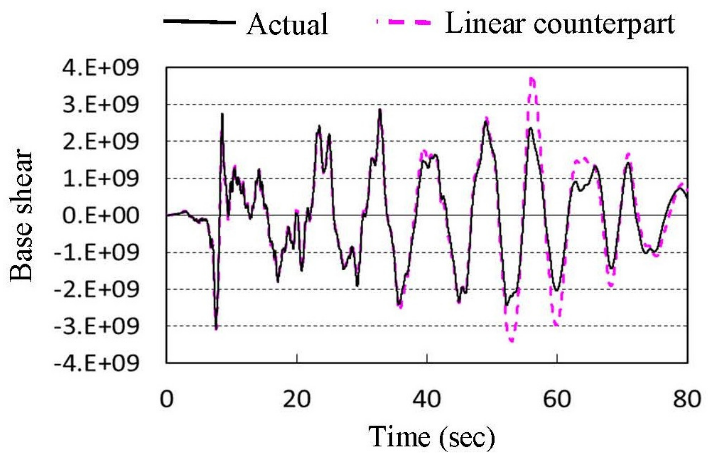
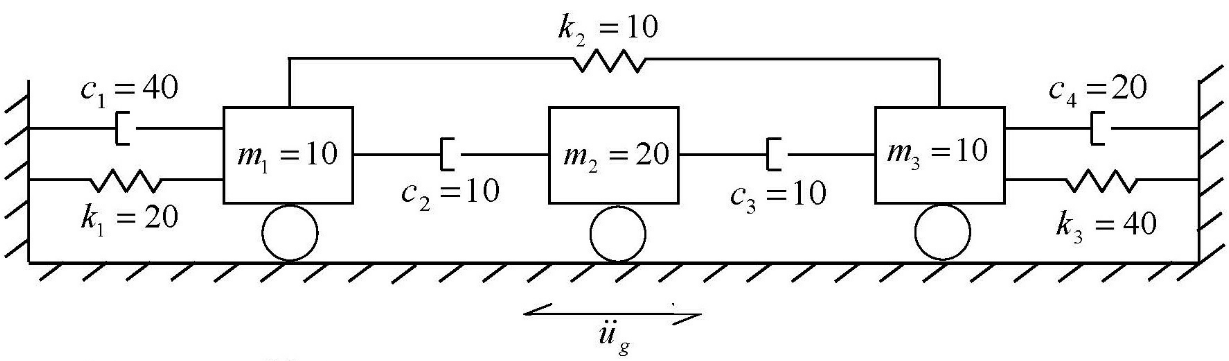
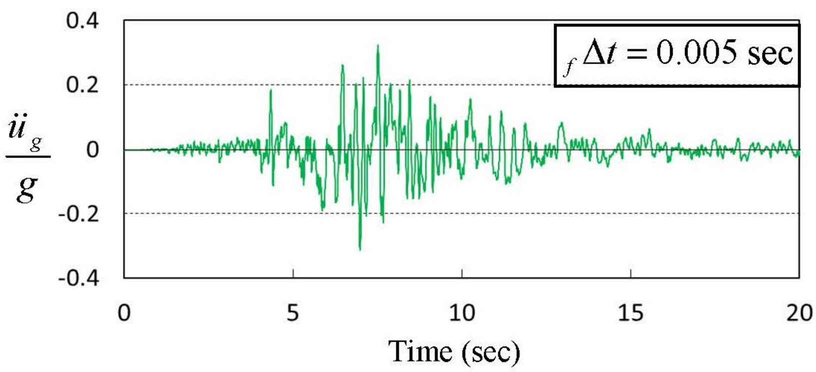


Figure 10. Exact history of the target response in the first example and its linear counterpart.

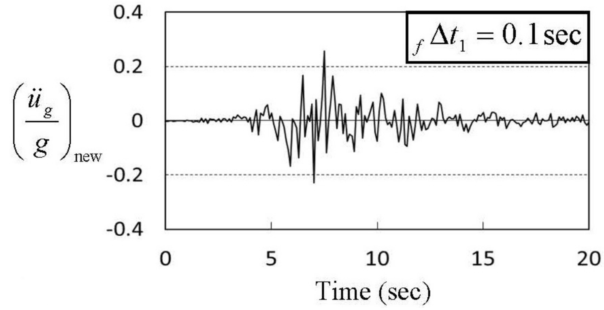


(a). Structure's model.

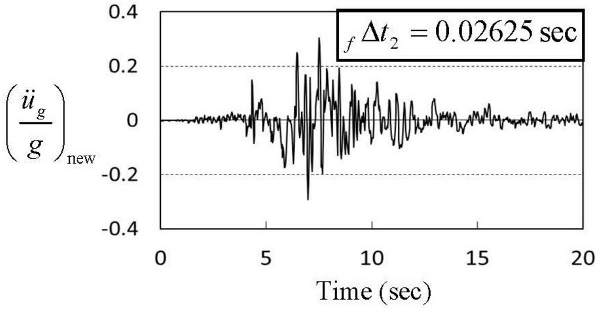


(b). Ground acceleration.

Figure 11. The structural system in the second example.

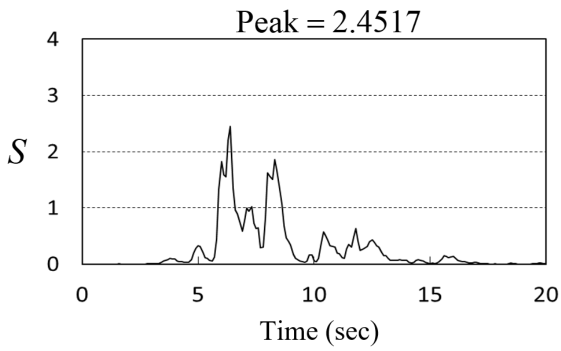


(A). $j = 1$.

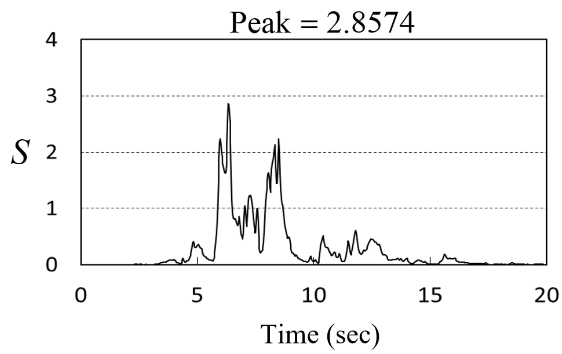


(B). $j = 2$.

Figure 12. Records obtained from Steps (e) and (f) of the proposed procedure for the second example.

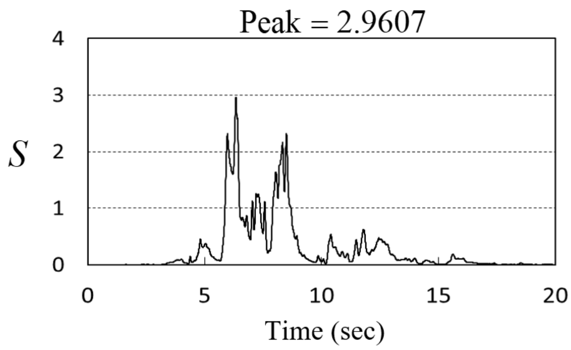


(A). $j = 1$.

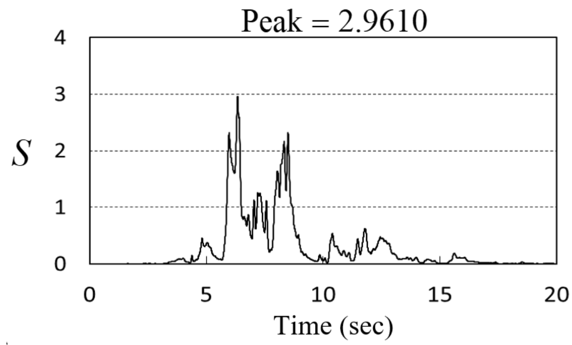


(B). $j = 2$.

Figure 13. History of the target response and the peak value obtained from Steps (d) and (e) of the proposed procedure applied to the second example.



(A). $\Delta t = {}_f\Delta t$.

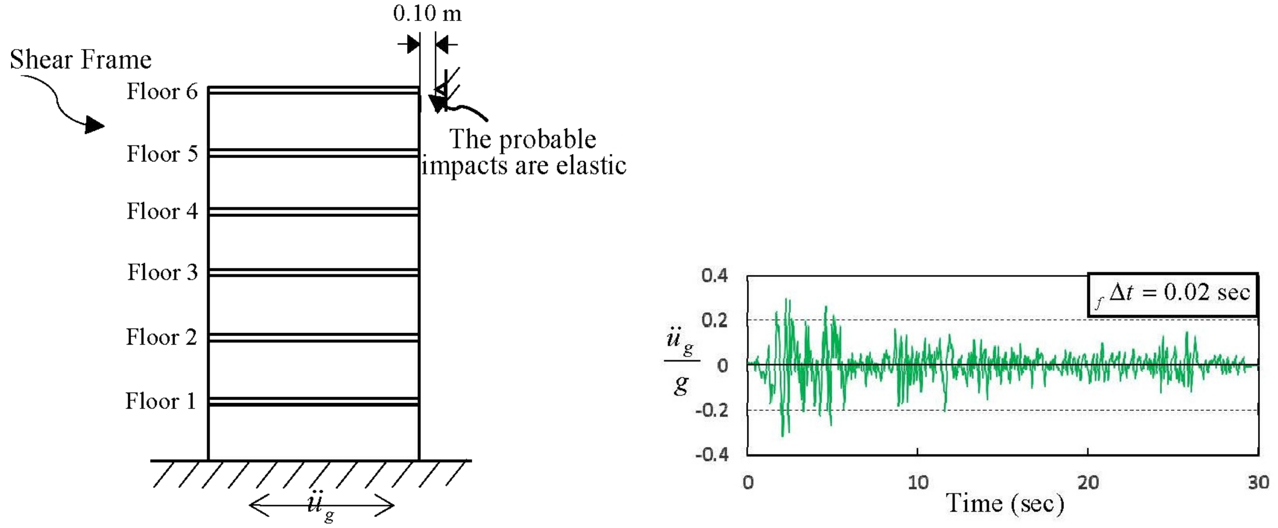


(B). $\Delta t = 0.5\,{}_f\Delta t$.

Figure 14. History of the target response together with the peak value obtained from ordinary time history analysis of the second example.

method is set to the HHT method [101] ($\alpha = -0.05$, $\gamma = 0.55$, $\beta = 0.275625$). As the results of Steps (b)–(d), $j = 1$ and $n_1 = 20$. The result of Steps (e) and (f) is the digitised record displayed in Figure 12a. Steps (g) and (h) lead to the target response history and the peak value shown in Figure 13a. Since only one time-integration computation is carried out, the computation proceeds to Step (c), which, together with Step (d), lead to $j = 2$ and $n_2 = 10.5$. Using these results, Steps (e) and (f) lead to Figure 12b, from which Steps (g) and (h) lead to Figure 13b. The peak target responses in Figures 13a and 13b are compared with respect to Equation (23), and as the result, the response reported in Figure 13b is final.

The first time integration computation of the ordinary time history analysis is carried out with steps equal to that of the excitation record, i.e. ${}_f\Delta t = 0.005\,\mathrm{s}$. The obtained target response is shown in Figure 14a. The time integration is then repeated with half a step, resulting in Figure 14b. The peaks of the two responses are within 0.02 % relative difference. Accordingly, Figure 14b represents the final target response. The accuracy is determined by comparing Figures 14b and 13b, and the analysis run-times are compared by the number of steps, which is clear due to the linear behaviour. As a result, by using the SEB THAAT according to the proposed procedure, assigning a value to n is auto-

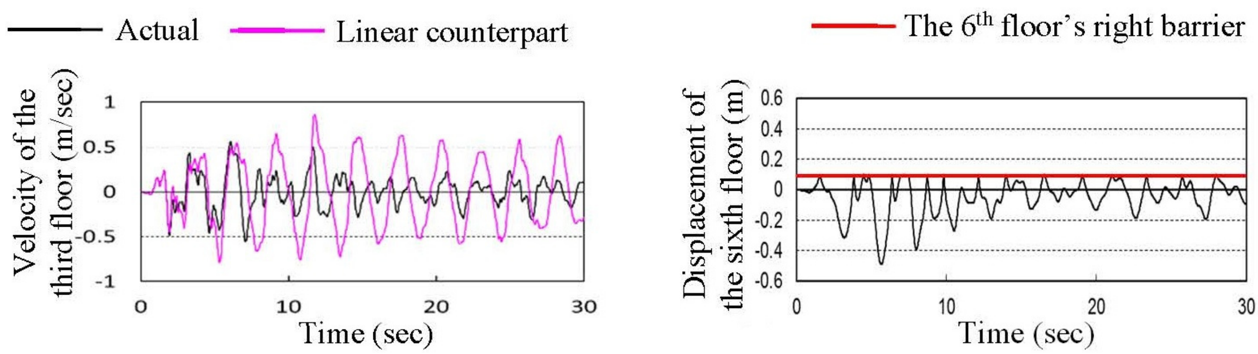


(A). Model of the structure. (B). Ground acceleration.

Figure 15. The structural system in the third example [96].

Property	**Floor (i)** 1	2	3	4	5	6
$10^{-9} \times m_i$ [Kg]	1.8	1.8	1.8	1.8	0.6	0.6
$10^{-11} \times k_i$ [N/m]	1.2	1.2	1.2	1.2	0.2	0.2
Damping [N s m^{-1}]	Classical [33], considering 2 % damping for the 1st and 3rd natural modes					

Table 7. Main properties of the structural system in the third example [96].



(A). Target response. (B). Displacement of the sixth floor.

Figure 16. Exact responses of the system introduced in Figure 15 and Table 7.

mated, the response is computed accurately, and the run-time is reduced, with the decrease of the number of integration steps from 12 000 ($\frac{20}{0.005} + \frac{20}{0.005\times0.5}$) to 962 ($\frac{20}{0.005\times20} + \frac{20}{0.005\times0.5\times10.5}$), i.e. 91.98 %.

4.4. Examples three and four: Tests on analyses non-linear due to elastic impact

4.4.1. A brief overview

Pounding and collision are of major causes of destruction during earthquakes [102–105]. Besides, time history analysis is a powerful tool for seismic analysis (see [1, 33, 35]) and, in the first and second examples, the damping was nonzero and non-classical. Considering these, two structural models, involved in elastic impact, one damped classically, and one undamped, are studied, in the next two subsections.

4.4.2. A classically damped system involved in elastic impact

Consider the structural system introduced in Figure 15 and Table 7. The average acceleration method [106] is selected for the time integration, and the velocity of the third floor is set as the target response. Given Figure 16, the elastic impacts actually occur and cause the behaviour to be non-linear. For both the ordinary and proposed analyses, details of the non-linear solution are set similar to those in the first example. After the sequential time integration computations, the results shown in Table 8 and Figure 17 are achieved.

Features	Analysis	
	Ordinary	According to the proposed procedure
Number of time integration computations	2	2
Total number of integration steps	4 788	888
Decrease in the analysis run-time [%]	81.45	

Table 8. Summary of the analysis run-time study for the system introduced in Figure 15 and Table 7.

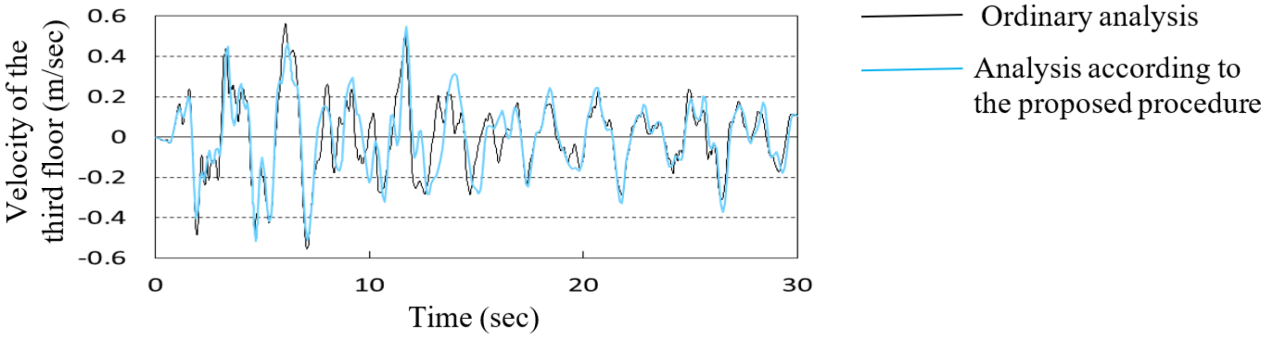


Figure 17. Final responses obtained for the system introduced in Figure 15 and Table 7 using average acceleration time integration method.

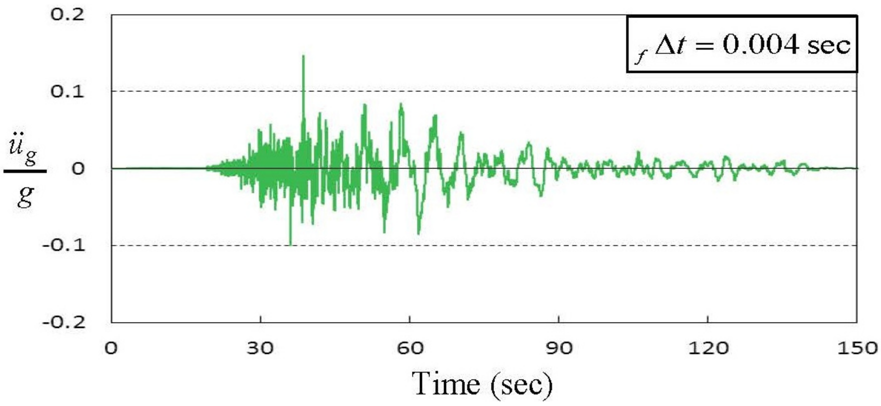


Figure 18. The ground acceleration in repetition of the study of the system introduced in Figure 15a and Table 7.

Accordingly, the reduction in the analysis run-time is 81.45 %, and despite the slight difference between the two graphs in Figure 17, the accuracy is acceptable, considering that:

(1.) The main seismic features of the response [33] are not changed,

(2.) both the ordinary and the proposed analyses do not consider the accuracy in the entire history,

(3.) neither of the two graphs in Figure 17 are exact,

(4.) results of non-linear dynamic analyses are rarely exact [1, 89, 97, 107].

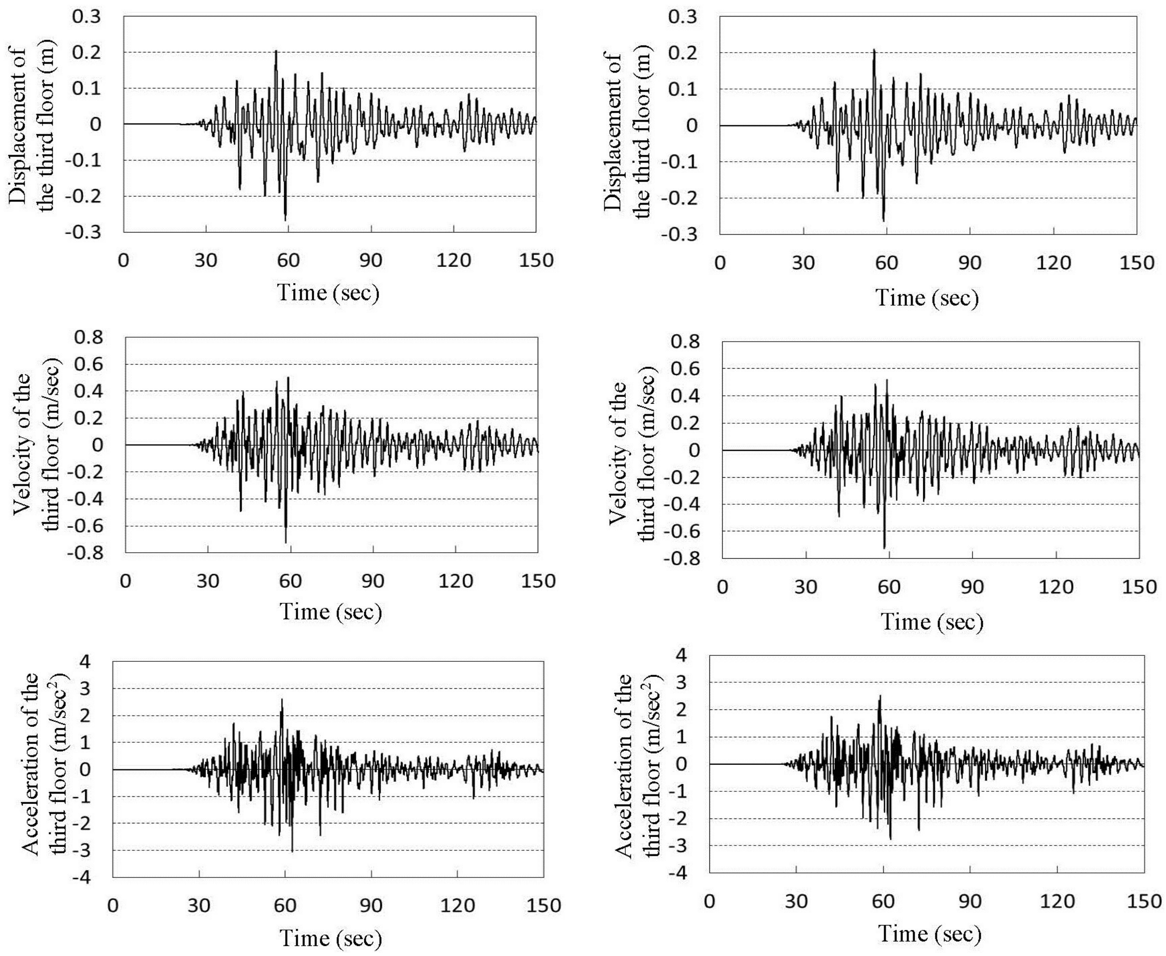
The study is repeated, considering the excitation in Figure 18 (instead of that in Figure 15b). Also, instead of one target response, three target responses, including the third floor's displacement, velocity, and acceleration, are taken into account, separately. Results of the study are summarised in Table 9 and Figure 19, displaying the very good performance of the SEB THAAT when applied according to the proposed procedure. In more detail, while the difference between Figures 19a and 19b is negligible, the computation leading to Figure 19b is about 10.8 times faster, i.e. the analysis run-time is 90.71 % shorter. Table 9 and Figure 19 also display that the performance of the proposed procedure is not necessarily sensitive to the target response. In contrary, comparing Table 8 with Table 9 and Figure 17 with Figure 19 implies that the performance of the SEB THAAT's application using the proposed procedure can be sensitive to excitation. Finally, Figure 20 confirms non-linearity of the dynamic behaviour.

4.4.3. An undamped system involved with elastic impact and material non-linearity

The structural system in this section is the bridge structure introduced in Figure 21 (see [96, 108]), where the vertical movements are neglected, the decks are rigid, the piers are massless, the damping is zero, the

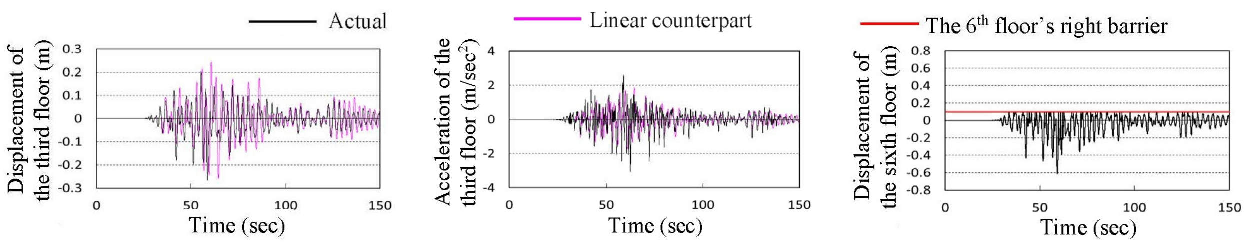
Target response	Number of time integration computations		Total number of integration steps		Decrease in the analysis run-time [%]
	Ordinary analysis	Analysis according to the proposed procedure	Ordinary analysis	Analysis according to the proposed procedure	
3^{rd} floor's displacement	2	2	113 056	10 500	90.71
3^{rd} floor's velocity	2	2	113 056	10 500	90.71
3^{rd} floor's acceleration	2	2	113 056	10 500	90.71

TABLE 9. Summary of the analysis run-time study for the system introduced in Figures 15a and 18 and Table 7.



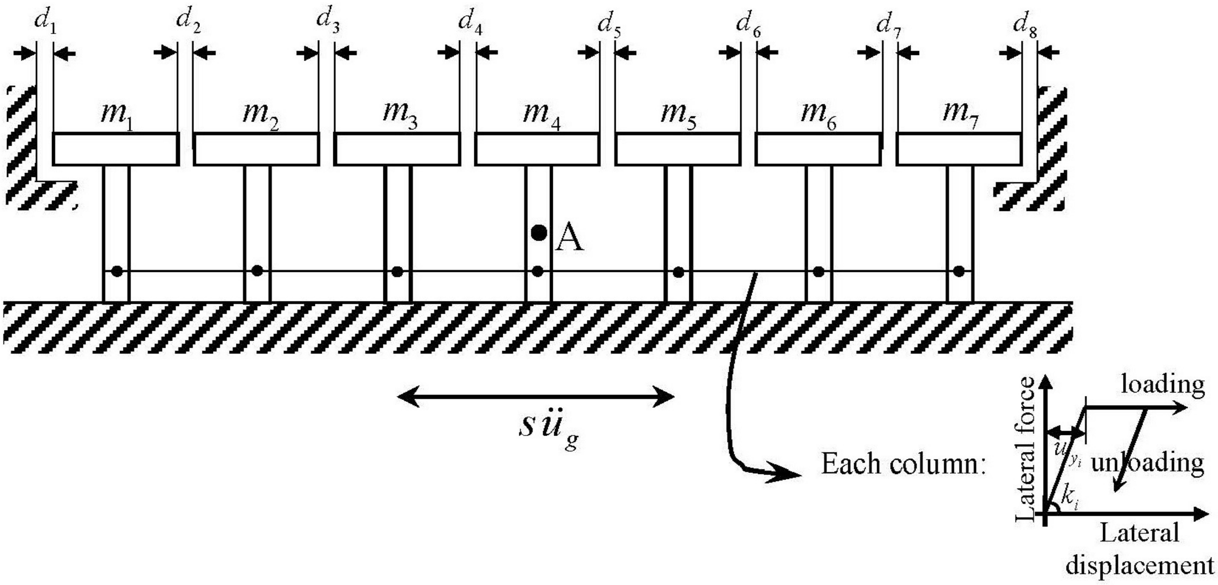
(A). Analysed ordinarily. (B). Analysed according to the proposed procedure.

FIGURE 19. Final responses for the system introduced in Figures 15a and 18 and Table 7 using the average acceleration time integration method.

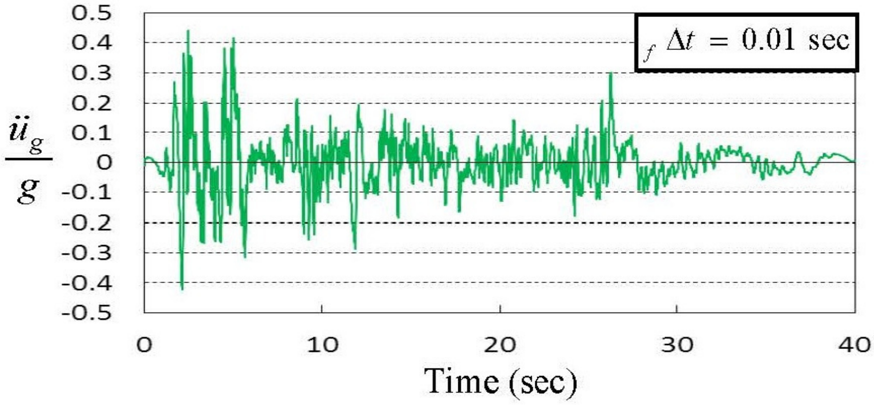


(A). The first target response. (B). The third target response. (C). Displacement of the sixth floor.

FIGURE 20. Evidences for the non-linear behaviour of the system introduced in Figures 15a and 18 and Table 7.



(a). Model of the structure.



(b). Ground acceleration.

Figure 21. The structural system in the fourth example.

impacts are elastic, and the following three alternatives are considered for the s in Figure 21a:

$$s = 0.4657588964,\ 0.7452142342,\ 1.117821351, \quad (30)$$

and the rest of the parameters are set as follows:

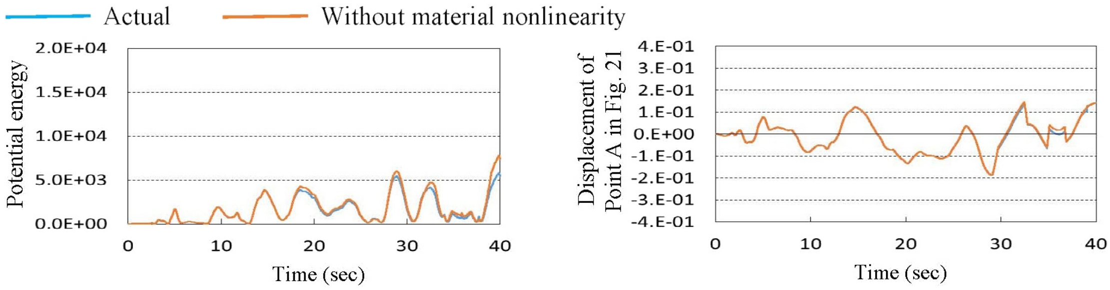
$$d_i = 0.2\,\mathrm{m}, \quad k_i = 0.2 \times 10^5\,\mathrm{N\,m^{-1}},$$
$$m_i = 3 \times 10^6\,\mathrm{kg}, \quad u_{y_i} = 0.3\,\mathrm{m}, \quad (31)$$
$$i = 1, 2, 3, \ldots, 7.$$

Equation (30) introduces three cases, where the structural system has 25 %, 100 %, and 200 % additional excitation, compared to when the system is in the vicinity of linear/non-linear behaviour, for which, $s = 0.3726071171$ (note that the non-linearity is of piece-wise linear type [1, 107, 108]). These percentages can also be referred to as the severity of the non-linear behaviour, SN; see [96]. Accordingly, by studying the performance of the proposed procedure for different values of s, we can arrive at an idea about the effect of severity of the non-linear behaviour on the performance. The displacement of Point A in Figure 21 (the central pier's mid-point), u_{A}, and the potential energy of the system, E_P, i.e.:

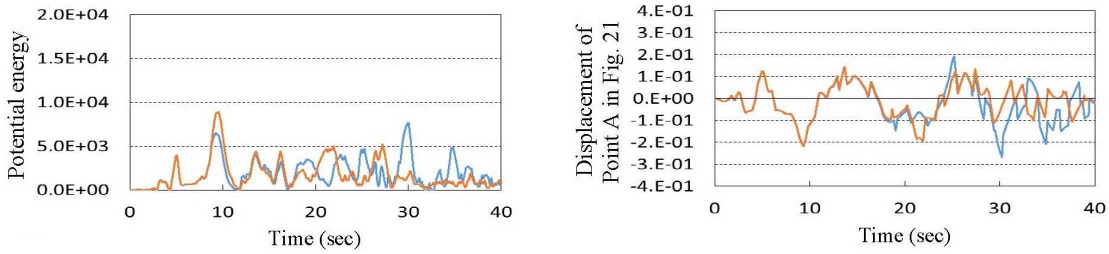
$$E_P = \frac{1}{2} \sum_{i=1}^{7} u_i f_i, \quad (32)$$

are considered as target responses, in two separate studies (the new variable f_i stands for the shear force of the i^{th} pier from left). In addition to the fact that, because of Equation (30), the behaviour is non-linear, given the sizes of d_i and u_{y_i} in Equation (31), the elastic impact is always involved in the non-linear behaviour. By removing the material non-linearity, the target responses change as shown in Figure 22, in orange. Evidently, the contribution of material non-linearity is negligible when SN = 25 % and is significant when SN = 100 % and SN = 200 %. Meanwhile, by comparing the blue graphs in Figures 22a, 22b, and 22c, we can get an idea of the extent to which the non-linearity can affect the behaviour.

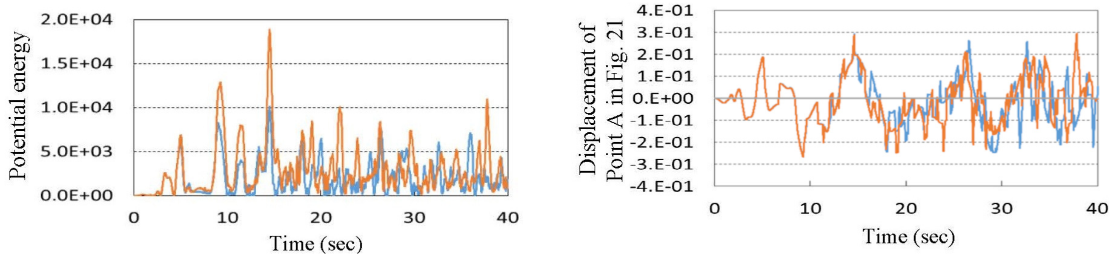
In addition to changes in the physical parameters, i.e. the target response and the severity of non-linear



(a). SN = 25 %.



(b). SN = 100 %.



(c). SN = 200 %.

Figure 22. Exact target responses of the system defined in Figure 21 and Equation (31).

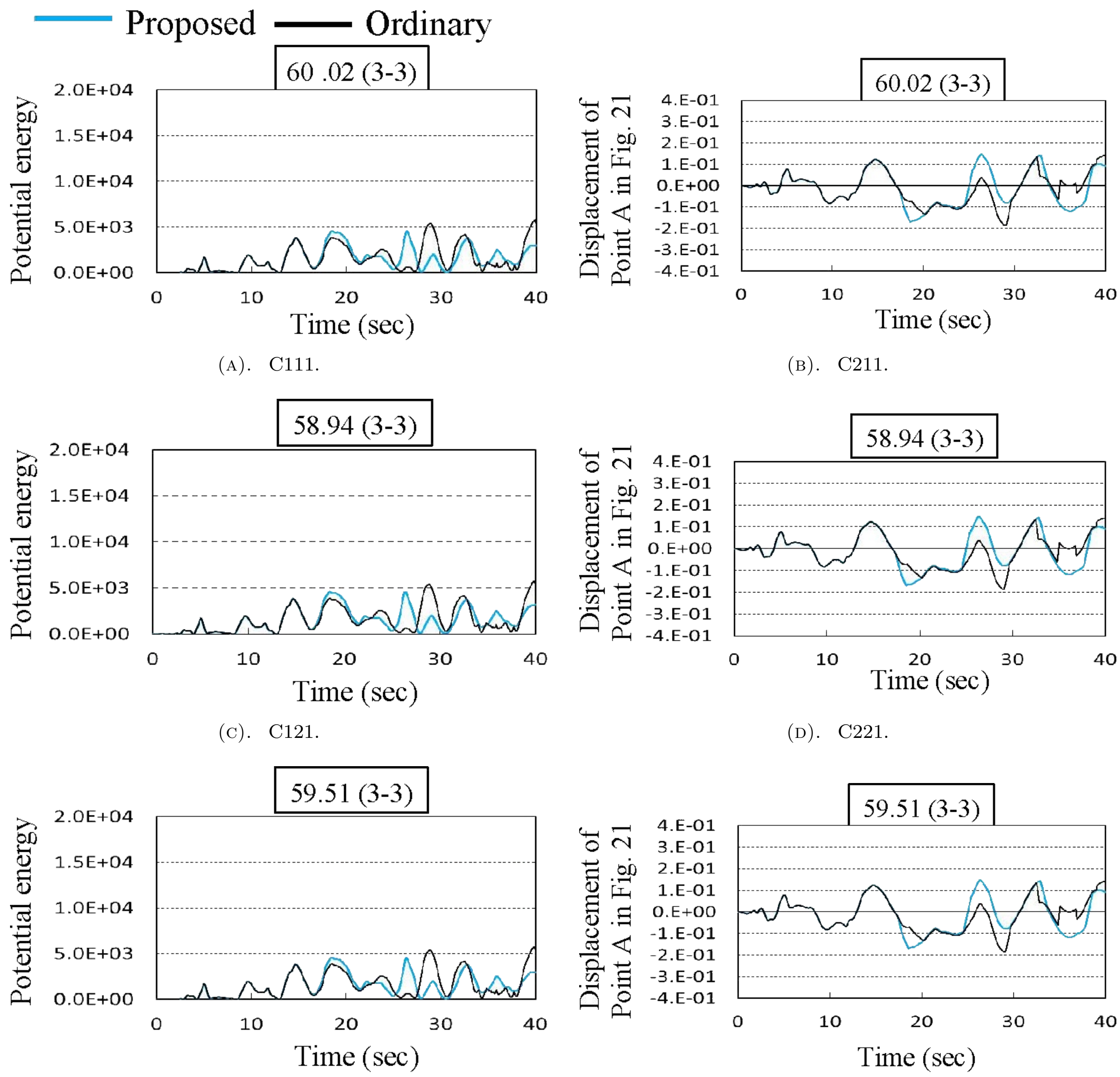
Case	Target response	Integration method	SN [%]	Case	Target response	Integration method	SN [%]
C111	u_A	Average Acceleration	25	C211	E_P	Average Acceleration	25
C112	u_A	Average Acceleration	100	C212	E_P	Average Acceleration	100
C113	u_A	Average Acceleration	200	C213	E_P	Average Acceleration	200
C121	u_A	Central difference	25	C221	E_P	Central difference	25
C122	u_A	Central difference	100	C222	E_P	Central difference	100
C123	u_A	Central difference	200	C223	E_P	Central difference	200
C131	u_A	C-H ($\rho_\infty = 0.85$)	25	C231	E_P	C-H ($\rho_\infty = 0.85$)	25
C132	u_A	C-H ($\rho_\infty = 0.85$)	100	C232	E_P	C-H ($\rho_\infty = 0.85$)	100
C133	u_A	C-H ($\rho_\infty = 0.85$)	200	C233	E_P	C-H ($\rho_\infty = 0.85$)	200

Table 10. The eighteen cases under consideration in the study of the system introduced in Figure 21 and Equation (31).

behaviour SN, changes in the integration method (as a computational parameter) are also taken into account. The analyses are carried out thrice, using the average acceleration [106], the central difference [109], and the C-H [100] ($\rho_\infty = 0.85$) time integration methods. Consequently, eighteen cases are included in the study; see Table 10, where, in the first and fifth columns, the three digits in right of each C introduce the target response, the integration method, and the severity of non-linear behaviour, respectively. The final results are reported in Figures 23–25, where the first number in the top boxes implies the percentage of reduction in the analysis run-time and the two numbers in the parentheses are the numbers of repetitions in the ordinary and proposed analyses, respectively. The main observations are as follows:

(1.) The final target responses obtained from the proposed analysis approach and ordinary analysis do not always match over the entire analysis interval.

(2.) In rare cases, the application of the SEB THAAT using the proposed approach seems to slightly increase the analysis run-time; see Figures 25a and 25e.



(a). C111. (b). C211.

(c). C121. (d). C221.

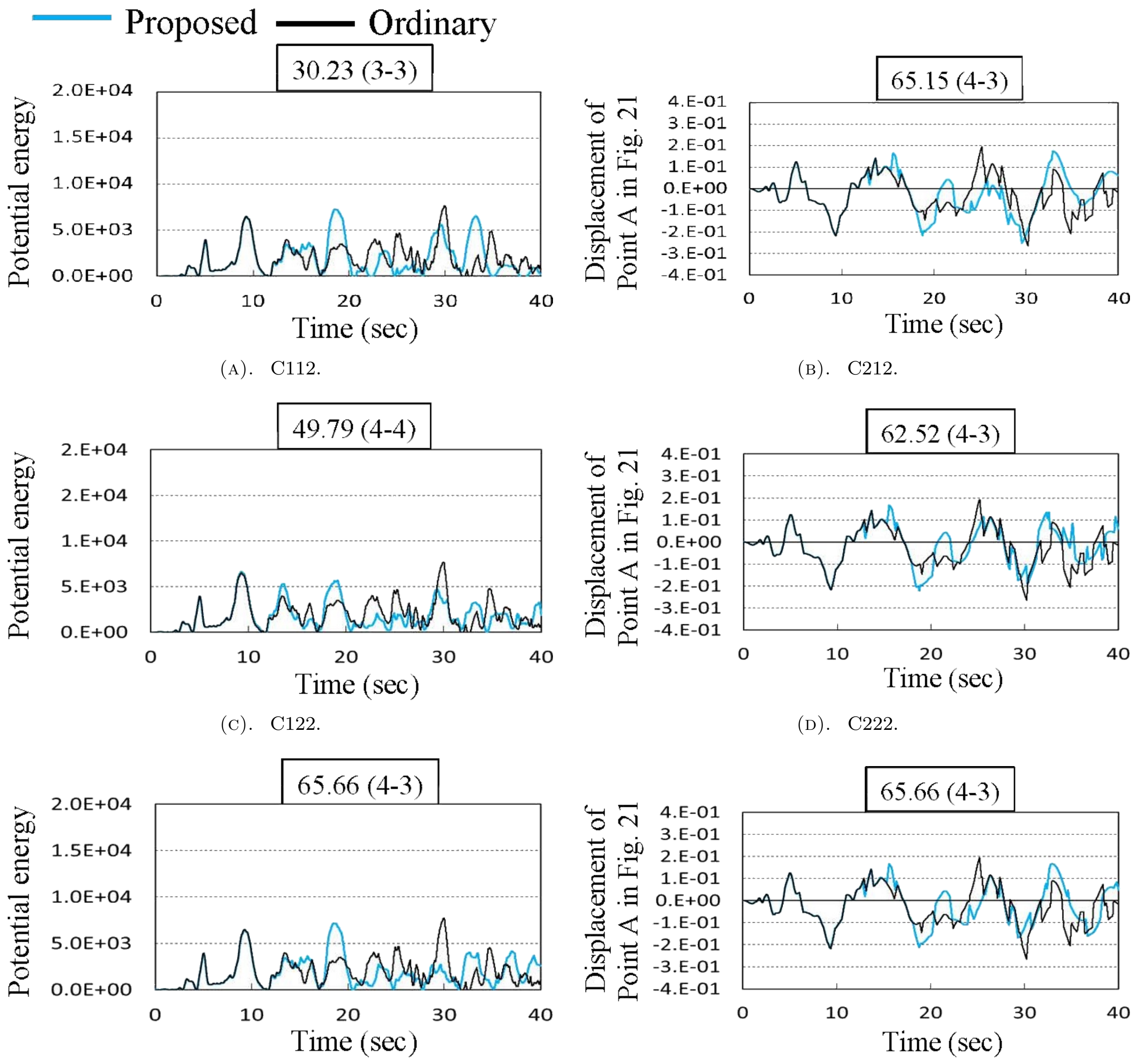
(e). C131. (f). C231.

Figure 23. Final responses obtained for the systems introduced in Figure 21 and Equation (31) and the corresponding reductions in the analysis run-times, when SN = 25 %.

(3.) Both the accuracy of the target response and the reduction in the analysis run-time because of applying the SEB THAAT according to the proposed procedure may be sensitive to the target response, the integration method, and the severity of non-linear behaviour. In particular, the reductions in the run-time are generally greater when examining the displacement of Point A. Meanwhile, the sensitivities are greater when the SN is greater.

The first observation can be explained by the seismic requirements that influence the analysis procedures (see also [33, 35, 92]). Based on these needs, the ending criterion of both the ordinary and proposed analyses is the convergence of the peak target response. Therefore, it is not reasonable to expect the two responses to necessarily match or even come close to each other over the entire time frame of the analyses. Accordingly, the accuracy shown in Figures 23–25 is acceptable. Further explanation is presented in Section 6.

Regarding the second observation, firstly, the number of cases displaying a longer analysis run-time when using the proposed procedure is low and the observed amount of the increase is small. Then, the observation is in agreement with the literature [1, 46, 79], according to which, the details of the non-linear solution should be set carefully. For instance, for this specific problem, by changing the maximum number of non-linear iterations from five to three, the reduction of the analysis run-time in Figure 25a changes from −9.07 % to approximately +14.25 %. Thirdly, as stated in Section 4.1, the first time integration computation of ordinary analyses is generally carried out with a step obtained from Equation (2) or a slightly different



(a). C112. (b). C212.

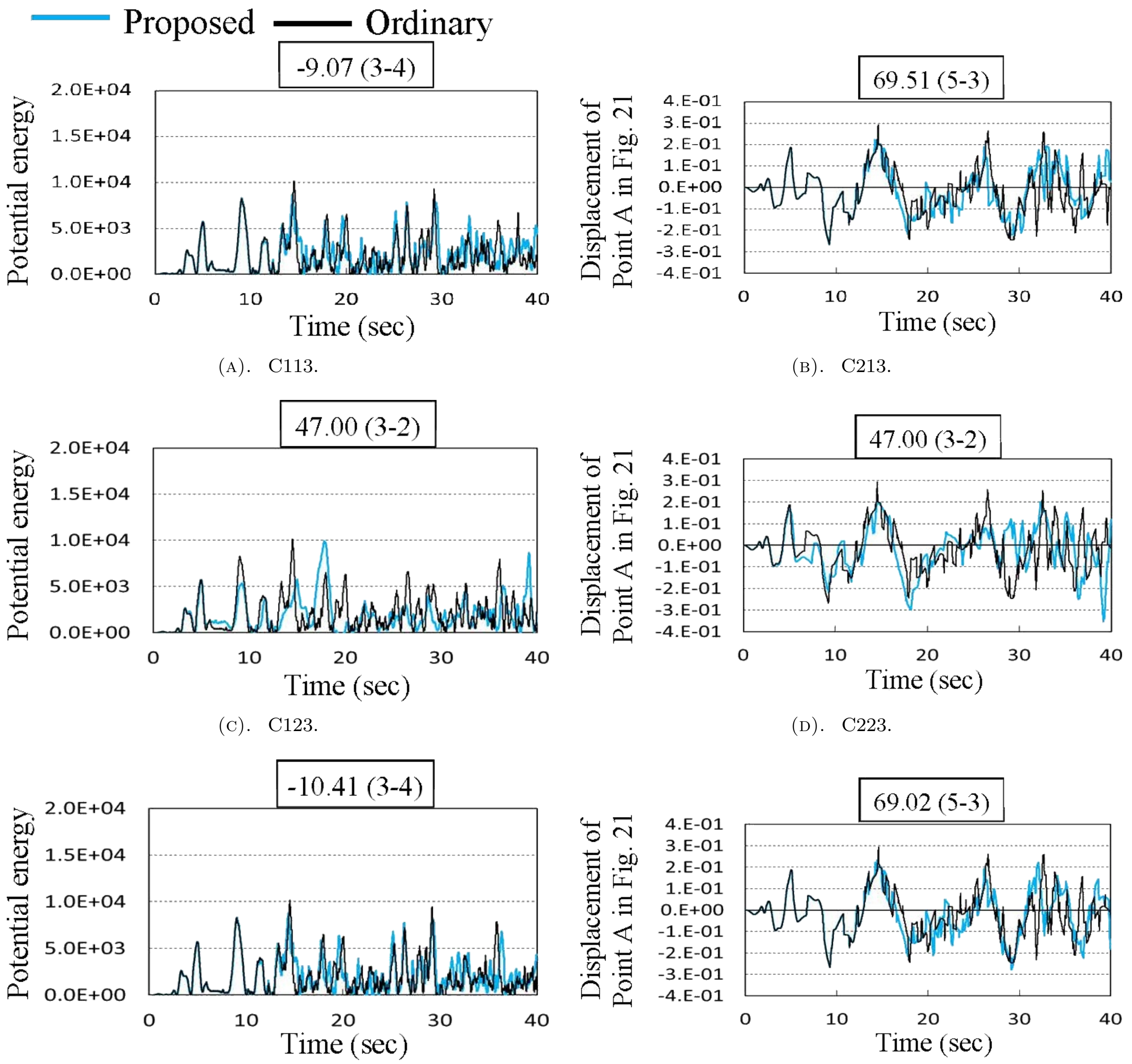
(c). C122. (d). C222.

(e). C132. (f). C232.

Figure 24. Final responses obtained for the systems introduced in Figure 21 and Equation (31) and the corresponding reductions in the analysis run-times, when SN = 100 %.

version of Equation (2) (e.g. see [35, 92]). For the previous examples in this paper, the result of Equation (2) is not very different from ${}_f\Delta t$. In this example, however, the result of Equation (2) is about 30 times smaller than ${}_f\Delta t$. Using the correct step for the first time integration computation of the ordinary analysis changes Figure 25 to Figure 26, where the reduction in the analysis run-time is considerable and about that of the previous examples, and the accuracies of the blue graphs are about those of the black graphs, as well as the blue graphs in Figure 25. Finally, and in completion of the previous explanation, the behaviour of the system corresponding to Figure 25 is very complicated; Figure 27 clearly displays that, when SN = 200 %, the response is not only highly oscillatory [110, 111], but also mathematically stiff [112, 113].

And to explain the third observation, the time history analysis, ordinarily or according to the proposed procedure, is sensitive to the target response (because of the ending criterion of the analysis), is sensitive to the integration method (because of the direct effect of the integration method on the response), and is sensitive to the SN (because changes in the SN change the excitation). Therefore, it is reasonable to expect that the difference of the final responses obtained from the two analyses also depend on these three parameters and the performance of using the SEB THAAT according to the proposed procedure to be sensitive to these parameters. It is also worth noting that because of the fact that convergence is preserved in different stages of the discussion, and the ending criterion in the proposed procedure is in close relation to the convergence (see Appendix A), the sensitivity to the integration method is much lower than the sensitivities to the



(a). C113. (b). C213.

(c). C123. (d). C223.

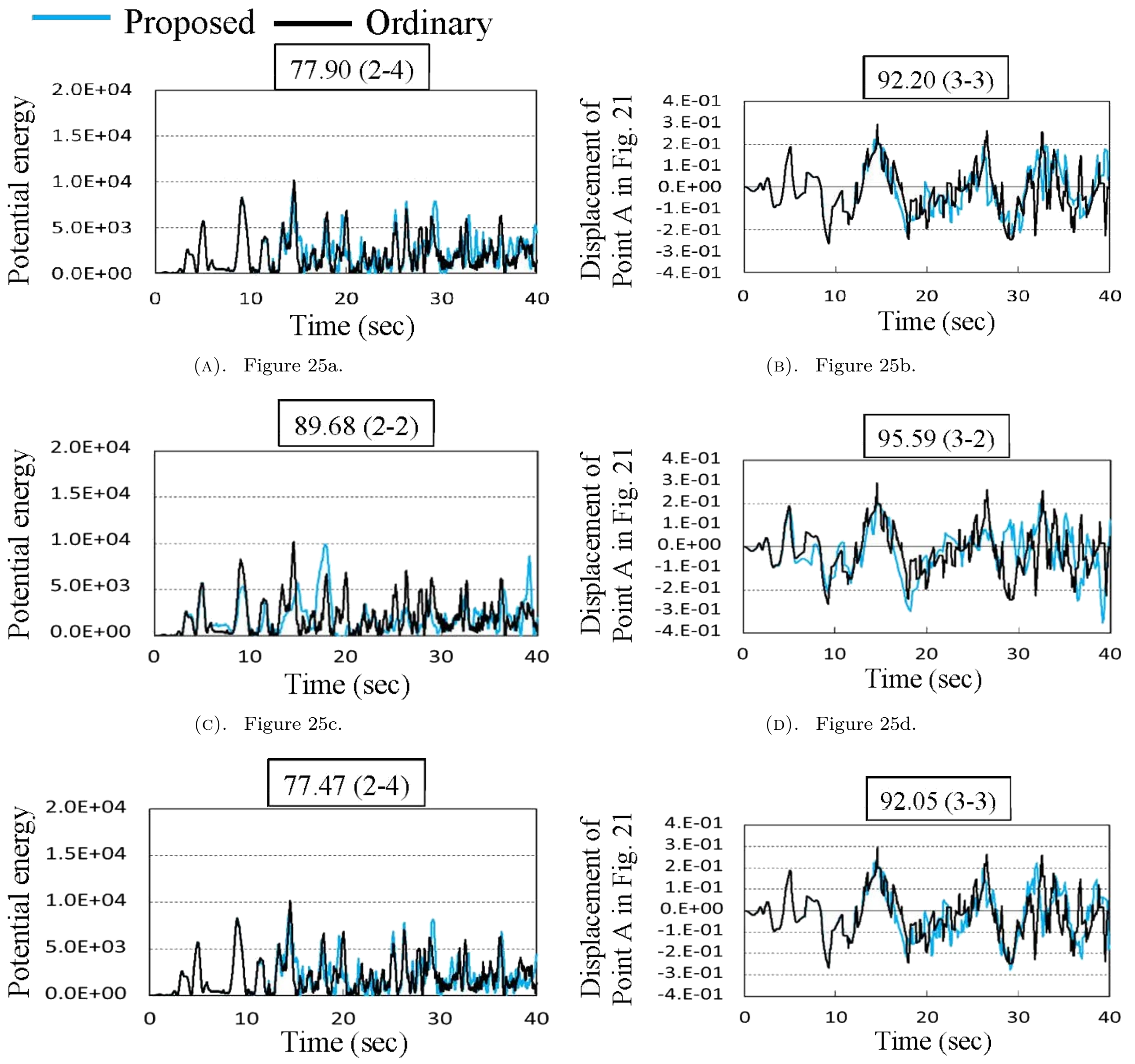
(e). C133. (f). C233.

Figure 25. Final responses obtained for the systems introduced in Figure 21 and Equation (31) and the corresponding reductions in the analysis run-times, when SN = 200 %.

SN and target response. In more detail, sensitivity to the integration method is negligible, unless when the SN is very high and proper convergence is delayed to smaller steps; see Figures 23–25 and [1, 89, 97]. The better performance when the target response is the displacement of Point A can be explained by the fact that both the SEB THAAT and the proposed procedure are based on the convergence of the responses; see also [1, 39]. Non-linear behaviour potentially conflicts with convergence of the responses produced by the time integration [97]. Besides, because of the nature of the problem, specifically the similarity of the column characteristics and the structure's geometry, the most crucial source of non-linearity is the collision of the first and last decks with the adjacent supports. Given this, the location of Point A in the structural system, and that the potential energy is affected by the response at different locations in the system, the effect of the non-linearity on the displacement of Point A is less than the effect on the total potential energy. Consequently, the performance is reasonably better for the displacement of Point A. The lower sensitivity at lower SN values can be explained in a similar way.

5. A realistic example

Most of real structural systems are of a few thousand degrees of freedom. Therefore, to display that the proposed procedure can be successful in practice, a three-dimensional steel structure subjected to a two-component earthquake is studied in this section; see Figures 28 and 29 and Table 11. When modelling the structural system, no specific assumption, e.g. shear building assumption, is taken into account. One node is added at the mid-point of each beam and column,



(A). Figure 25a. (B). Figure 25b.

(C). Figure 25c. (D). Figure 25d.

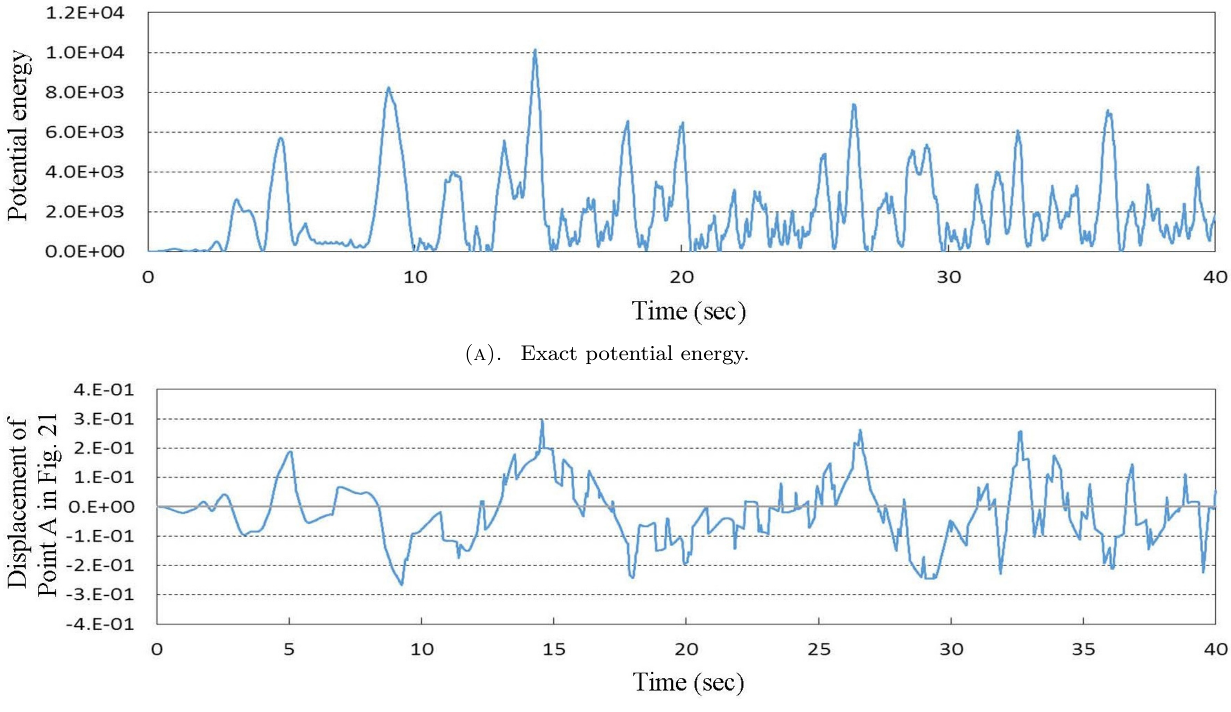
(E). Figure 25e. (F). Figure 25f.

Figure 26. Changes due to the use of the correct value of the integration step in the first time integration of the ordinary analyses.

with exception of the beams already halved by bracings, as well as the beams at the highest level. Each of the four beams at the highest level is divided to six beam-elements. The lengths of beam-elements are hence equal to two meters throughout the model. This results in a model with 4 968 degrees of freedom. The pattern of the bracings is continued to the ground. The lumped masses are placed at the beam-column connections, equal to 5 000, 10 000, 20 000, and 43 200 kg at the corners of each level of the structure (not at the top level), at the periphery of each level (not at the corners and top level), at the connections not at the periphery, and at the connections at the top of the structure (see Figure 28a), respectively. Damping is assumed to be of Rayleigh type [33], and equal to five percent in the first and third natural modes of the linear structure. Four target responses are taken into account simultaneously; given Figure 28a, acceleration and displacement of Point A in the x and z directions, respectively, displacement of Point B in the x direction, and the total base shear. The latter is obtained from:

$$V_{\mathrm{BS}} = \sqrt{\left(\sum_{\text{Base Columns}} R_x\right)^2 + \left(\sum_{\text{Base Columns}} R_y\right)^2}, \quad (33)$$

where, V_{BS} represents the total horizontal force transmitted to the foundation (disregarding the damping forces), and R_x and R_y stand for the shear force at the lowest floor's typical column in the x and y direction, respectively. Figure 28b, together with the difference between Figures 30a and 30b, confirms that the structural behaviour is non-linear. Simple two-node beam-column elements and two-node truss elements are used for the finite element modelling of the beams (and columns) and bracings, respectively [4, 75, 94]. The



(a). Exact potential energy.

(b). Exact displacement of Point A.

Figure 27. Complexity of the structural behaviour in the fourth example when SN = 200 %.

average acceleration method [106] is used for the time integration. The Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [114, 115] is used for the non-linear solution. Finally, the OPEN System for Earthquake Engineering Simulation (OPENSEES) is chosen as the structural analysis software [116].

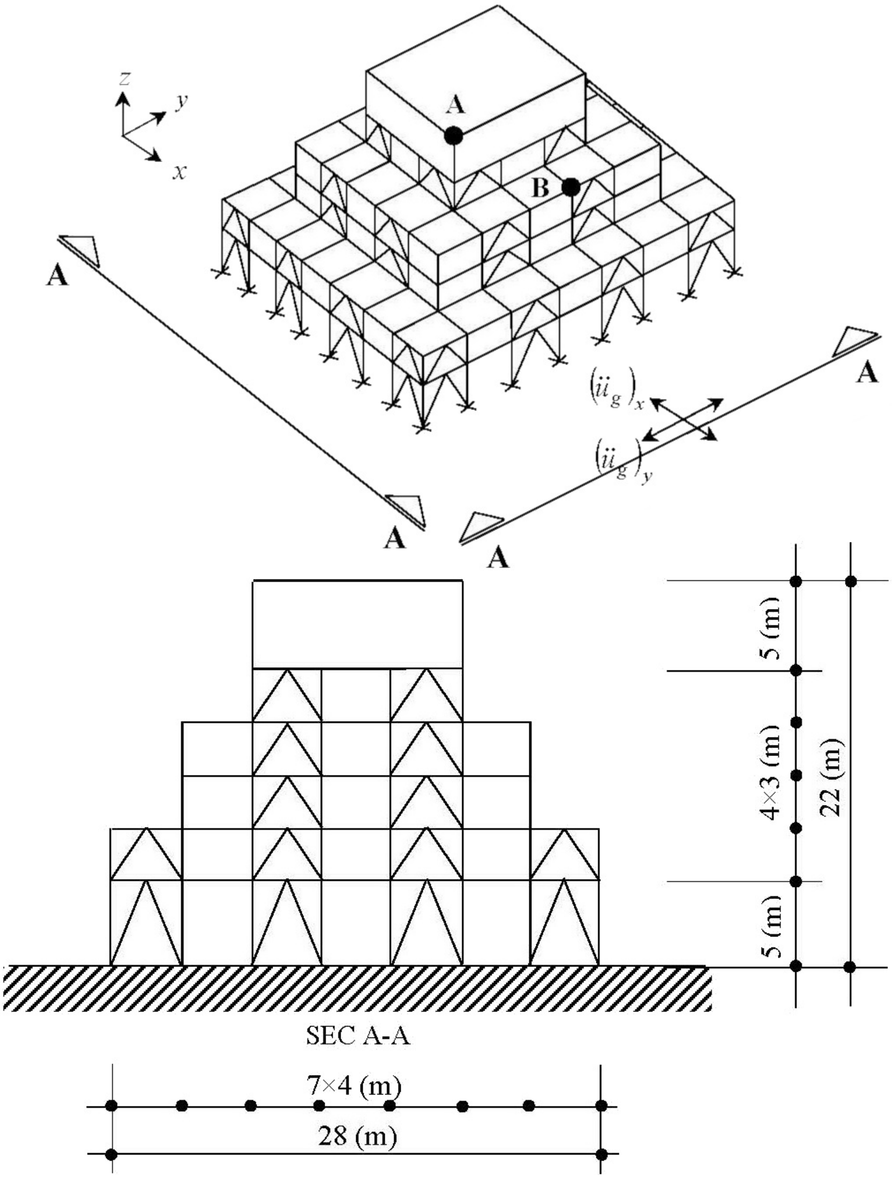
The results of analysing the system ordinarily and when applying the SEB THAAT using the proposed procedure are reported in Table 12 and Figure 31. Accordingly, the performance of the proposed procedure may be acceptable for realistic structural systems. The increase in computational efficiency is considerable, as well. Finally, the presented example differs from the previous examples in the size of the structure, the consideration of a two-component earthquake as the excitation, the non-linear solution method, and the consideration of multiple target responses simultaneously.

6. Discussion

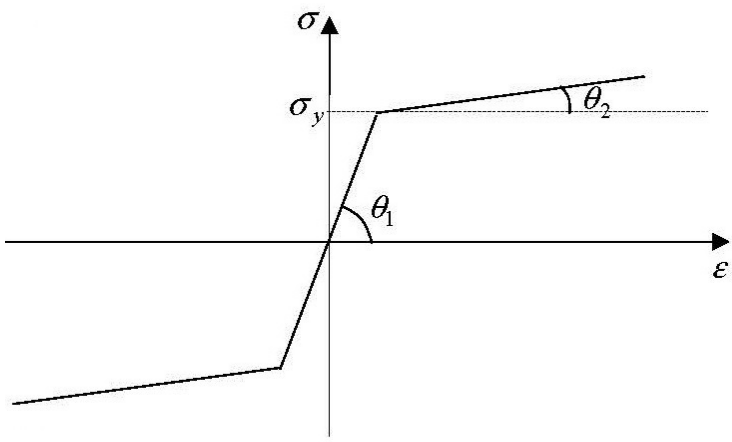
6.1. The achievements and their significance

In line with the objective of the paper, the main achievement is that the SEB THAAT can now be applied, without assigning a value to the enlargement scale n. This is important from a practical point of view, particularly because of the significant variation in the reduction in the analysis run-time, reported in Table 1.

Meanwhile, we can compare the previous applications of the SEB THAAT, summarised in Table 1, with the 25 observations reported in Sections 4 and 5 (see Table 13). As a result, when using the proposed procedure, the overall reductions in run-time are higher and the sensitivity to the problem is lower. In addition, compared to the previous tests, which were some on linear and some on non-linear analyses, the tests reported in this paper (with the exception of the second example) are on non-linear analyses, some with complicated behaviour. In conclusion, the use of the proposed procedure for application of the SEB THAAT, can be considered adequate. For a detailed comparison, the first example, which showed an 87 % reduction in the analysis run-time at the cost of a visually unrecognisable change in accuracy (see Table 13), is re-examined using the SEB THAAT without the new procedure. For this example, the application of the SEB THAAT considering different values of the enlargement scale n, the integration method and the target response according to Section 4.2, and a conventional value for the non-linear tolerance, i.e. 1.E−4 (see [117]), leads to Figure 32. It can be clearly seen that without using the proposed procedure, the 87 % reduction in the analysis run-time, observed in Section 4.2, is achieved at the cost of 33 % change in the accuracy of the target response when using $n = 36$. In comparison, the use of the SEB THAAT according to the proposed procedure, achieved only a 1.21 % change in accuracy, as can be seen from Figures 8b and 9b, and its performance is, therefore, superior to the previous applications of the SEB THAAT. Accordingly, using the proposed procedure for application of the SEB THAAT in the first example has considerably increased the computational efficiency of the



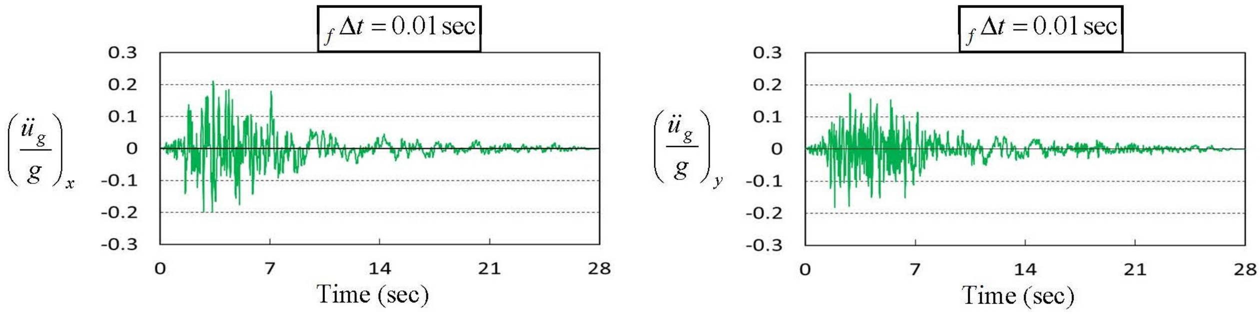
(A). Model appearance.



Notation	Explanation
ε	Strain
σ	Stress
$\tan\theta_1$	Young's modulus equal to 2.1×10^{11} N/m^2
$\tan\theta_2$	$0.02\tan\theta_1$
σ_y	(Yield stress) 2.353×10^{8} N/m^2
Hardening	Kinematic
Poisson Ratio	0.3

(B). Material.

Figure 28. Structural system in the realistic example.

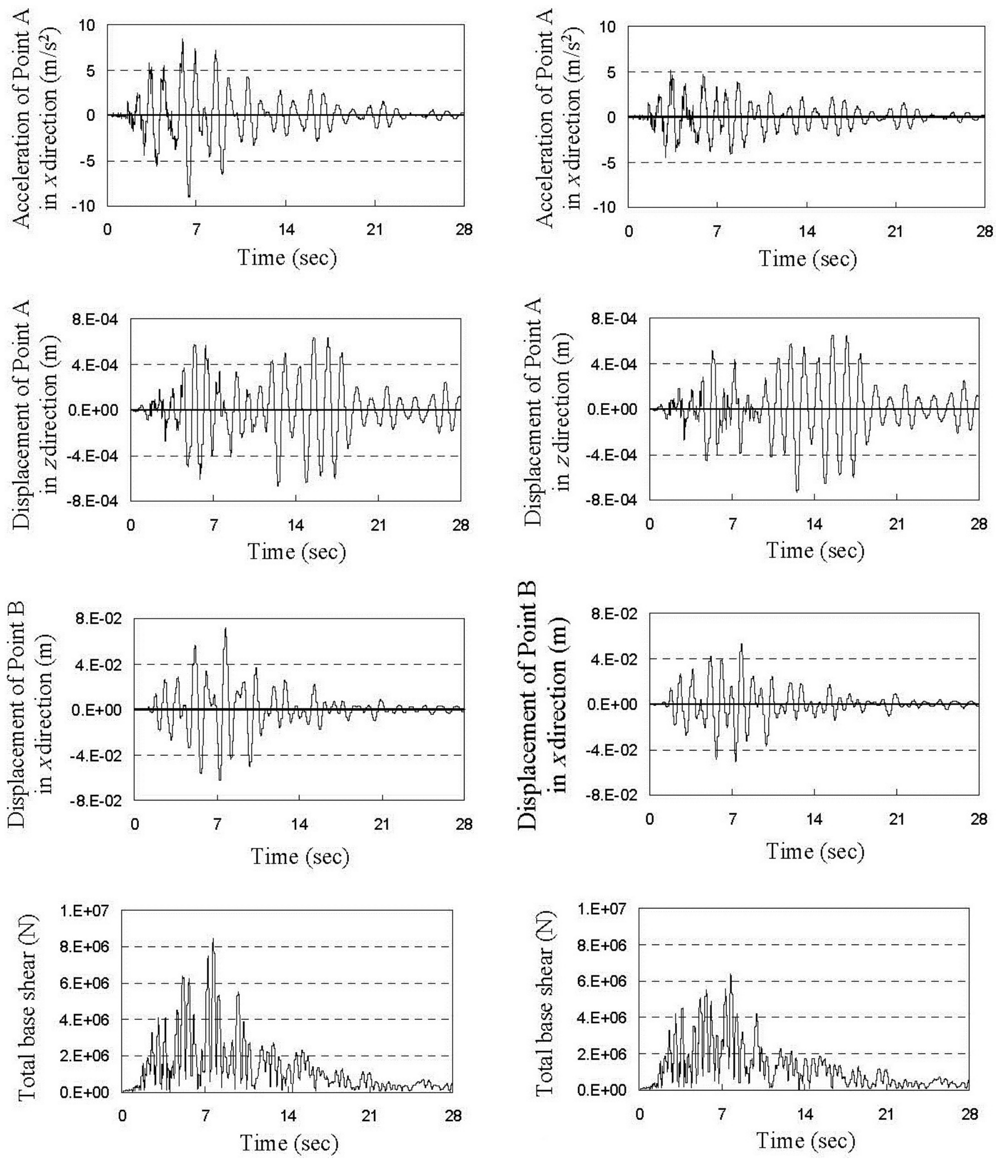


(A). x direction component. (B). y direction component.

Figure 29. Earthquake record in the realistic example.

Structural member	Profile	Area [cm^2]	Moment of inertia [cm^4]
Beams	IPE 270	45.90	5790
Columns	Box 360×360×16	214	41 450
Bracings	2U160	48.00	-

Table 11. Structural members and their properties in the realistic example.



(A). The linear counterpart. (B). The actual system.

Figure 30. Target responses of the realistic example.

Analysis	Number of time integration computations	Total analysis run-time	Reduction in the run-time [%]
Ordinary	2	$12^{h}\ 56'\ 11''$	93
Using the SEB THAAT according to the proposed procedure	2	$56'\ 23''$	

TABLE 12. Summary of the analysis run-time study for the system in the realistic example.

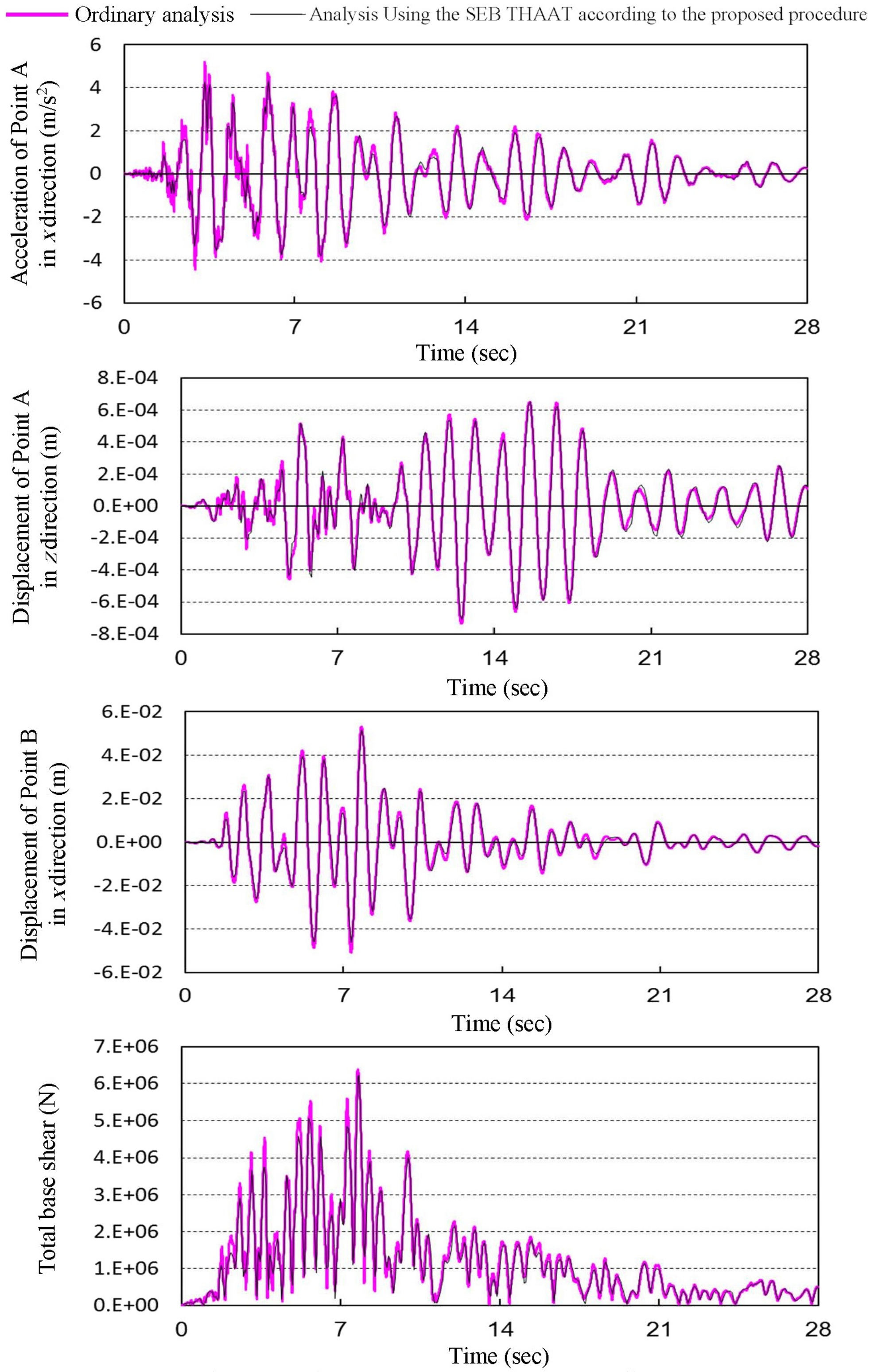


FIGURE 31. Accuracy of the proposed approach in the realistic example.

Structural system	Reduction in the analysis run-time [%]	Accuracy of the target response	Details
An eighteen story building	87	Visually matching in the entire analysis interval	See Section 4.2
A linear three-DOF system	92	Visually matching in the entire analysis interval	See Section 4.3
A six story building subjected to pounding at the sixth floor studied in four cases	81–91	Visually matching almost in the entire analysis interval	See Subsection 4.4.2
A multi-span bridge in 18 different cases of non-linear behavior, integration method, and target response	77–97*	The peak target responses, and the frequency content are very close	See Subsection 4.4.3
A realistic six floor building	93	Visually matching almost in the entire analysis interval	See Section 5

* For the sake of brevity, only six results are presented in Figure 26; however, the presented reduction is based on 18 results.

Table 13. Brief review of the reduction in analysis run-times in Sections 4 and 5.

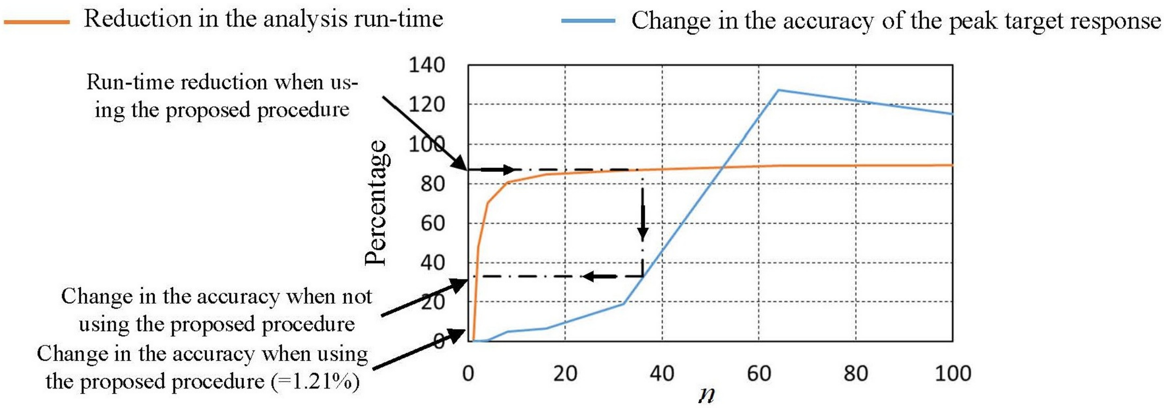


Figure 32. Changes in analysis run-time and accuracy of the peak target response in terms of n when applying the SEB THAAT to the analysis of the first example without using the proposed procedure.

SEB THAAT. Consequently, the proposed procedure, besides eliminating the ambiguity of assigning correct values to the enlargement scale n, may enhance the computational efficiency of using the SEB THAAT. This implies improvement in simplicity and efficiency of the SEB THAAT.

In addition, unlike previous applications of the SEB THAAT, the parameters T and χ do not play a role in the application of the SEB THAAT using the new procedure. In other words, whereas in the previous applications, the T and χ were considered approximately and ambiguously but directly, by using the proposed procedure, the roles of T and χ are implied in the repetitions of time integration computation indirectly but clearly and with less approximation. This removes the problems of assigning values to T and χ, and makes the application of the SEB THAAT to be more simpler and clearer.

Due to the capabilities of time integration in the analysis of systems with static instability (zero stiffness) [1, 2, 23], it is reasonable to expect the SEB THAAT to perform well in these analyses. This has not yet been investigated [38], and the second example in this paper is indeed the first report in this field. Finally, none of the previous applications of the SEB THAAT consider multiple target responses simultaneously; the realistic example presented in this paper is a pioneer in this area.

6.2. The weak points

Parameter-less simple and clear application of the SEB THAAT with acceptable reduction in the analysis run-time, and accuracy of the target response, is the main advantage of the proposed procedure compared to the previous applications of the SEB THAAT. However, there are also disadvantages.

A disadvantage of the proposed procedure is that the final target response may be digitised in steps which, if smaller than the step of the excitation $_f\Delta t$, may not be a fraction of $_f\Delta t$, and, if larger than $_f\Delta t$, may not be a multiplier of $_f\Delta t$. This is trivial in theory, but inconvenient in practice, and can be a hindrance in post-processing. For example, it may prevent a simple increase of the accuracy of the responses by Richardson extrapolation [25, 26, 118].

A second disadvantage is that using the proposed procedure does not guarantee a decrease in analysis run-time. In other words, although, in view of the examples presented, the analysis run-time decreases when the SEB THAAT is applied according to the proposed procedure, and the probability of the run-times increase decreases when the details of the non-linear solution are well set, the probability has not been theoretically prevented. This disadvantage is likely to be more pronounced if the problem and/or the analysis is unusually complex.

Finally, inherited from the New Zealand Seismic Code, NZS 1170.5: 2004 [35, 92], when applying the SEB THAAT according to the proposed procedure, the analysis ends with convergence of the peak target response. This criterion may be inappropriate for important analyses, especially when the target response is to be computed accurately in the entire analysis interval.

6.3. More on the complicatedness of the bridge example

As implied in [96], for the structural system introduced in Figure 21 and Equations (30) and (31), both the structural behaviour and the time integration computation are complicated. The main reasons are:

(1.) The non-linearity of the system originates in two different sources, i.e. elastic collision and material non-linearity; see Figure 22 and the following fourth point.

(2.) Because of Equation (30), the behaviour is complicated, especially when SN = 200 %; see Figures 22c and 27.

(3.) Also because of the shape of the excitation in Figure 21b, the mathematical stiffness [112, 113] of the problem is considerable.

(4.) As the main reason for complexity of the behaviour, in view of Figure 21a and Equation (31), the collisions and material loading/unloading at different locations of the structural system and at different time instants are completely dependent. In other words, independent of the excitation and the SN, the non-linearity starts with the collision of the first or seventh mass to the neighbouring support. Then, when the columns are in the plastic range, the collision of the mass above the column with one of the two neighbouring masses (or support) will usually result in unloading and change in the behaviour of the column from plastic to elastic. This means that many collisions and unloading occur simultaneously, which adds complexity to both the behaviour and the computation. Finally, the complexity because of this reason increases by the second and third reasons mentioned above, i.e. high SN and mathematical stiffness.

(5.) The last example in [96] differs from the bridge example presented in Subsection 4.4.3, in the earthquake excitation and the values of SN. The two excitations in the example in [96] are not particularly more complex than the excitation in Subsection 4.4.3. In addition, while the structural behavior in the example presented in [96] is very complicated, the maximum values of SN in the examples in [96] and this paper are 100 % and 200 %, respectively. Therefore, the bridge example presented in this paper is more complicated than that presented in [96].

The complexity of the behaviour and the computation leads to a weaker performance of the proposed procedure in this special example as compared to the other examples studied in this paper; see Table 13. Nevertheless, this is consistent with the nature of time integration analyses, which may require very small integration steps to achieve sufficient accuracy in non-linear complicated analyses; see [1, 86–90, 96, 97, 107, 108, 119, 120]. In other words, the weaker performance in highly complicated non-linear problems, though would rather be eliminated, is to be expected, taking into account the performance of well-known time integration methods in analysis of complex structural dynamic behaviour [2, 23, 86–90, 121–126].

6.4. Comparison with other analysis acceleration methods

The aim of this paper is only to simplify the application of the SEB THAAT by eliminating the parameters from the application process. The objective has been achieved and, in the cases studied (as examples), efficiency has been improved; see Table 13. In addition, the SEB THAAT has already been compared with some other techniques; see [1, 42]. Taking this into account, the comparison presented in [1] is briefly extended in Table 14 to a rough comparison between the following eight techniques:

(1.) The SEB THAAT according to the new procedure, proposed in this paper,

(2.) direct down sampling [28],

(3.) parareal time integration [20–22, 127],

(4.) time integration of integrated problems [41],

(5.) combination of truncation and direct down sampling [30],

(6.) impact based replacement of the earthquake record [40],

(7.) the SEB THAAT by assigning values to the enlargement scale n [38, 39],

(8.) adaptive fast non-linear analysis; see [128–130].

The comparison shows the superior performance of the SEB THAAT using the proposed procedure from various points of view. It is also worth noting that the overall reduction in the analysis run-time because of the eighth technique [128–130] is slightly higher than that of the first technique; see Table 13 and [128].

Feature	Technique (1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Simplicity of application	a	b	e	d	d	d	c	d
Effect on the in-core memory	a	a	c	b	a	a	a	b
Number of the past successful tests	b	a	c	d	c	d	b	c
Theoretical basis	a	c	a	d	c	c	b	b
Versatility	a	a	a	b	a	b	a	c
Capability to be simply plugged in analysis software	a	a	d	e	b	c	a	c
Using all the excitation data	a	c	a	a	b	a	a	a
Computational effort needed for application	b	a	f	g	d	e	b	c
Additional computational facilities needed	b	a	e	d	c	c	b	c
Number of parameters additional to ordinary analysis	a	b	d	a	c	b	b	b
Applicability beyond structural dynamics	a	a	a	b	b	c	a	c

* The “a”, “b”, “c”, “d”, “e”, “f”, and “g”, respectively, imply that the technique is the first, second, third, fourth, fifth, sixth, and seventh best technique from the point of view of the feature stated in the first column.

Table 14. A rough comparison between eight techniques for accelerating time history analysis of Equation (1)*.

However, even from the point of view of computational efficiency, the first technique is superior due to its negligible impact on the in-core memory, and the larger number of the examples presented in this paper compared to those presented in [128, 129].

6.5. Main challenges and a perspective of the future

By using the proposed procedure, the SEB THAAT can be simply and clearly applied to structural dynamic analyses. Considering this, Tables 1 and 13, and the significance of computations' run-times [131], it is reasonable to use the proposed procedure for application of the SEB THAAT in real analyses. More efforts are nevertheless essential (some ongoing), in the following directions:

(1.) With regard to the SEB THAAT and its application:
- (a) Enhancement of the application to complex structural dynamic analyses, e.g. highly oscillatory non-linear analyses.
- (b) Application to wave propagation problems. The behaviour of many important oscillatory systems is a combination of structural dynamics and wave propagation. As indicated in Table 1 and Sections 4 and 5, the SEB THAAT has mostly been tested in application to structural dynamic problems. In wave propagation problems, the number of degrees of freedom is generally larger, the run-times are more, and the need to speed up the analysis is greater.
- (c) Application to the analysis of systems subjected to various excitations. Many important structural systems can be subjected to several digitised excitations simultaneously, e.g. off-shore platforms that are exposed to earthquake, wind, and sea wave. These problems tend to be large, non-linear, and complicated and time consuming to analyse. The application of the SEB THAAT to such problems can be complicated, especially when the difference between the digitisation steps is considerable. It is therefore essential to improve the SEB THAAT and the proposed procedure to accelerate such analyses.
- (d) Increasing the public attention and interest in the SEB THAAT, by:
  - i. Preparing a user-friendly internet webpage to convert $\mathbf{f}(t)$ to $\mathbf{f}_{\text{new}}(t)$.
  - ii. Preparing a user-friendly internet webpage to apply the proposed procedure.
  - iii. Testing the application of the SEB THAAT according to the proposed procedure in the analysis of large realistic systems with complex behaviour (with a real need to reduce the analysis run-time).

(2.) Regarding the proposed procedure:
- (a) Introducing a theoretical guarantee, if necessary together with modifications of the proposed procedure, regarding the reduction of the analysis run-time, without significant change in the accuracy, when applying the SEB THAAT to arbitrary time history analysis.
- (b) Making changes to the procedure so that either the time step at which the final target response is reported is an integer multiple of the excitation digitisation step, or the excitation step is an integer multiple of the target response output step.
- (c) Modification of the ending criterion of the proposed procedure, for cases where checking the peak of the target response is not sufficient, e.g. very important structural systems.

The future of the SEB THAAT's application using the proposed procedure is promising in view of presented discussions, Tables 1, 13, and 14, Equation (12), and the following two facts:

(1.) Due to improvements in recording instrumentation [48, 132, 133], the smallest available value of ${}_f\Delta t$ is in continuous decrease.

(2.) Due to the continuous improvements in structural optimisation [134–136], the ever-increasing variety of material properties [137, 138], and the growing importance of financial aspects, the general trend of structural system changes is towards lighter and less stiff structures. This leads to oscillations of the target response in larger periods.

Many of the above seven challenges will be overcome in the near future. In addition, the SEB THAAT and the proposed procedure will be integrated in commercial structural analysis software, which will allow many numerical tests to be carried out leading to further improvements. Various theoretical improvements can be expected, as well. Finally, given the mathematical basis of the SEB THAAT [1, 38, 39], the application of the SEB THAAT using the proposed procedure can be tested on problems other than Equation (1).

7. Conclusion

The SEB THAAT was proposed in 2008 as a technique to replace the digitised excitations with excitations digitised in larger steps, so that time history analyses can be accelerated at the cost of acceptable changes in accuracy. After many successful tests on the SEB THAAT, this paper proposes a procedure that eliminates the need to assign values to the main parameter of the SEB THAAT. Assigning values to some parameters of the analysis is eliminated, as well. The proposed procedure has roots in the New Zealand Seismic Code, NZS 1170.5:2004 [35, 92], the computational traditions in structural dynamics [33], and numerical solution of ordinary differential equations [91]. The main achievements are as follows:

(1.) The SEB THAAT can now be applied with no concern about the parameters n, T, and χ; this is even simpler than an ordinary (without application of the SEB THAAT) analysis according to NZS 1170.5:2004 [35, 92].

(2.) Using the proposed procedure, the SEB THAAT can be applied, regardless of the problem, the excitation, the integration method, and the non-linear solution details, i.e. no limitation exists for the application of the SEB THAAT according to the proposed procedure.

(3.) In view of the presented twenty-five cases, the performance of the SEB THAAT, when applied according to the proposed procedure, is satisfactory. However, it is weaker in the analysis of complicated, highly oscillatory, and highly non-linear structural dynamic systems. The weakness is consistent with the characteristics of time integration analysis of highly oscillatory highly non-linear systems.

(4.) The previous point is valid for the accuracy of the target response, as well as the analysis run-time.

(5.) Both the reduction in the analysis run-time and the accuracy of the target response are potentially sensitive to the problem, the severity of non-linear behaviour, the target response, the excitation, and the integration method. Given the convergence, the sensitivity to the integration method is less than the other sensitivities, unless the behaviour is very complicated.

(6.) Compared to the SEB THAAT's previous applications, the performance of the SEB THAAT when applied according to the proposed procedure seems less sensitive to the problem.

(7.) The application of the SEB THAAT according to the proposed procedure seems leading to more computational efficiency compared to the previous applications of the SEB THAAT.

(8.) Inherited from the features of time integration, the SEB THAAT can reduce the analysis run-time in analysis of statically unstable systems, with negligible effect on the accuracy of the target response.

(9.) The SEB THAAT can perform well when several target responses are under consideration simultaneously.

(10.) Compared to several other analysis acceleration methods, the application of the SEB THAAT using the proposed procedure is superior, in terms of simplicity of application, negligible effect on the in-core memory, significant reduction in analysis run-time, etc.

Based on the above results, the author can recommend using the SEB THAAT according to the proposed procedure for analysis of arbitrary structural dynamic systems subjected to excitations available in digitised format. Given the significant reduction in the analysis run-time reported in Table 13, and the fact that still only twenty-five cases have been tested for the proposed procedure, it is recommended that in real applications additional checks for accuracy be performed after the final response is obtained. Repeating the analysis with other integration methods, or using the SEB THAAT without the proposed procedure can be two alternatives. These checks will negatively affect the reduction in the analysis run-time, but are essential until sufficient testing of the proposed procedure.

Some remaining challenges are as follows:

(1.) Improving the proposed procedure for cases with complicated, highly oscillatory, highly non-linear behaviour.

(2.) Detailed study of the ending criterion of the proposed procedure for applications where controlling the peak response is not sufficient for the response accuracy.

(3.) Testing and improving the SEB THAAT and the proposed procedure for an analysis of wave propagation problems.

(4.) Testing and improving the SEB THAAT and the proposed procedure for an analysis of structural systems subjected to several excitations, digitised in steps, sized differently.

Finally, using the proposed procedure, the SEB THAAT is applicable to analysis of problems with governing equations different from Equation (1). The performance, i.e., the reduction in analysis runtime and the accuracy of the target response, should, however, be investigated. This can accelerate analysis of systems in different fields, and besides may increase the interest in the SEB THAAT.

List of symbols

a An auxiliary variable in determination of the new excitation by the SEB THAAT

b_k An auxiliary variable in determination of the new excitation by the SEB THAAT

c_i Viscous damping of the i^{th} damper in Figures 6a and 11a

C-H Chung-Hulbert (time integration method)

Cijk An indicator for the cases examined in Subsection 4.4.3, introduced in Table 10

d_i Distances between the decks and the supports, introduced in Figure 21a

E Computational error

E_K Kinetic energy

E_P Potential energy

Er_j Error of the peak target response obtained from the j^{th} time integration computation of the proposed procedure

f_i Shear force of the i^{th} column from left in Figure 21a

$\mathbf{f}_{\text{int}}$ Vector of internal forces of an MDOF structural system

$\mathbf{f}$ Vector of external forces of an MDOF structural system

$\mathbf{f}_{\text{new}}$ New excitation obtained from the SEB THAAT

$\mathbf{f}_{\text{int}_0}$ Vector of internal forces of an MDOF structural system at $t = 0$

$\mathbf{f}_{\text{int}_i}$ Vector of internal forces of an MDOF structural system at $t = t_i$

$\bar{\mathbf{f}}$ An auxiliary variable for determining the new excitation by the SEB THAAT

g Acceleration of gravity

$\mathbf{g}(t)$ A record digitised in step ${}_f\Delta t_j$, obtained from the main excitation record, $\mathbf{f}$, using linear interpolation, in the proposed procedure

HHT Hilber-Hughes-Taylor (time integration method)

IDA Incremental Dynamic Analysis

k_i Stiffness of the i^{th} spring in Figures 6a and 11a

L_p Interval of the integration steps, at which the results of the analysis converge properly

m_i i^{th} mass in Figures 6a and 11a

$\mathbf{M}$ Mass matrix

MDOF Multi-Degree-of-Freedom

n Step enlargement scale and the only parameter of the SEB THAAT, eliminated when using the proposed procedure

n' An auxiliary variable for determining the new excitation by the SEB THAAT

n_1 Value assigned to n in the proposed procedure, in the first time integration computation

n_j Value assigned to n in the proposed procedure, in the j^{th} time integration computation

$n_{\max}$ Largest value of n satisfying the accuracy-based comments on time integration step

P_j The peak target response in the j^{th} time integration computation of the proposed procedure

P_{exact} The exact peak target response

$\mathbf{Q}$ Constraints in the governing equation that distinguish non-linear behaviour from linear behaviour

q Rate of convergence, generally equal to the order of accuracy of the time integration method

q' Rate of convergence of the approximation in the excitation

R_x x direction component of the shear force in the typical column of the lowest floor of the realistic example, without considering the damping forces

R_y y direction component of the shear force in the typical column of the lowest floor of the realistic example, without considering the damping forces

$\mathbf{R}$ The target response obtained from the ordinary time history analysis

$\mathbf{R}_{\text{new}}$ The target response obtained using the SEB THAAT

s A scaling factor for the excitation in Subsection 4.4.3

S Target response in the second example, equal to the sum of the kinetic and potential energies

SEB THAAT The name of the technique, its application is simplified in this paper (abbreviated from Step-Enlargement-Based Time-History-Analysis-Acceleration-Technique)

SI The International System of Units (abbreviated from the French Le Système International d'Unités)

SN A measure for severity of the non-linear structural dynamic behaviour, and abbreviation of Severity of non-linear structural dynamic behaviour

t Time

t_{end} Duration of the dynamic behaviour and the time history analysis

t'_{end} Duration of the new excitation obtained from the SEB THAAT

t_i i^{th} time station of the time integration computation

T Smallest oscillatory period with a worthwhile contribution to the target response

$({}_f\Delta t)_{\text{new}}$ Digitisation step of the SEB THAAT's result

Δt Step of time integration computation

Δt_j Integration step in the j^{th} time integration computation in application of the SEB THAAT according to the proposed procedure

χ A downscaling factor in Equation (2), introduced in Equation (3)

Δt_{cr} Upper bound of the integration step due to the linear theory of numerical stability

Δt_{CFL} Upper bound of the integration step in wave propagation problems associated with spatial discretisation

${}_f\Delta t$ Digitisation step of the excitation

${}_f\Delta t_j$ The digitisation step of the excitation in the j^{th} time integration computation in the analysis according to the proposed procedure

$\mathbf{u}$ Displacement vector of an MDOF structural system

$\dot{\mathbf{u}}$ Velocity vector of an MDOF structural system

$\ddot{\mathbf{u}}$ Accelerations vector of an MDOF structural system

$\mathbf{u}_0$ Displacement vector of an MDOF structural system at $t = 0$

$\dot{\mathbf{u}}_0$ Velocity vector of an MDOF structural dynamic system at $t = 0$

$\ddot{\mathbf{u}}_0$ Acceleration vector of an MDOF structural dynamic system at $t = 0$

$\mathbf{u}_i$ Displacement vector of an MDOF structural dynamic system at $t = t_i$

$\dot{\mathbf{u}}_i$ Velocity vector of an MDOF structural dynamic system at $t = t_i$

$\ddot{\mathbf{u}}_i$ Acceleration vector of an MDOF structural dynamic system at $t = t_i$

u_A Displacement of central pier's mid-point in Subsection 4.4.3

$\ddot{u}_g$ Ground acceleration

$(\ddot{u}_g)_x$ x direction component of ground acceleration

$(\ddot{u}_g)_y$ y direction component of ground acceleration

u_{y_i} Yield displacement of the i^{th} spring in Figure 6a

u_1 Displacement of the first mass in the second and fourth examples

u_2 Displacement of the second mass in the second and fourth examples

u_3 Displacement of the third mass in the second and fourth example

u_4 Displacement of the fourth mass in the fourth example

u_5 Displacement of the fifth mass in the fourth example

u_6 Displacement of the sixth mass in the fourth example

u_7 Displacement of the seventh mass in the fourth example

$\dot{u}_1$ Velocity of the first mass in the second example

$\dot{u}_2$ Velocity of the second mass in the second example

$\dot{u}_3$ Velocity of the third mass in the second example

V_BS Total horizontal force transferred to the foundation in the realistic example disregarding the damping forces

Z^+ The set of positive integers

α One of the three parameters of the HHT time integration method

β One of the three parameters of the HHT time integration method

$\bar{\delta}_j$ The non-linear tolerance in the j^{th} time integration computation

ε Uniaxial strain

σ Uniaxial stress

σ_y Uniaxial yield stress

γ One of the three parameters of the HHT time integration method

Γ A vector which, if the $\mathbf{f}$ in Equation (1) originates in $\ddot{u}_g$, is essential in computation of $\mathbf{f}$

ρ_∞ Spectral radius of a time integration method at very large values of $\frac{\Delta t}{T}$

$\tan\theta_1$ Young's modulus

$\tan\theta_2$ Slope of the second line in the uniaxial stress-strain plot, for materials with bilinear behaviour and kinematic hardening

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Appendix A.

A proof of Equation (23) is given in this appendix. In time history analyses equipped with accuracy control (e.g. as addressed in the New Zealand Seismic Code [35, 92]), it is common to repeat the first time integration computation with another time integration computation using half steps. Then, the absolute relative difference between the two peaks (i.e. $\frac{|P_1-P_2|}{P_2}$) is compared with 0.05. If the relative difference is larger than 0.05, the last time integration computation and the comparison of the two peaks are repeated until the difference becomes equal or smaller than 0.05, i.e. the 0.05 criterion is satisfied.

From the other side, a proper convergence of the response implies that the log-log plot of the error versus the integration step is a line with positive slope (the positive slope is generally referred to as the convergence rate and is almost equal to the order of accuracy) [51]; see Figure 2. Therefore, when the response obtained from a time integration computation converges properly, by repeating the computation with half integration steps, the error of the computation decreases 2^q-fold (q is the integration method's order of accuracy). In addition, close to the exact response, responses converge properly [51] and the approach to the exact response is from one side [139]. Accordingly, from simple mathematics [55], and under the assumption of sufficiently small integration steps (equivalent to the assumption of sufficient closeness to the exact response), comparing the absolute relative difference, i.e. $\frac{|P_1-P_2|}{P_2}$, with 0.05, implies comparing the relative error of P_1 [52], i.e.:

$$Er_1 = \left| \frac{P_1 - P_{\text{exact}}}{P_{\text{exact}}} \right|, \tag{34}$$

with $0.05(2^q-1)^{-1}2^q$ (where, P_{exact} is the exact peak response), which is the same as comparing the relative error of P_2 (the peak response of the analysis carried out with the smaller steps Δt_2), i.e.:

$$Er_2 = \left| \frac{P_2 - P_{\text{exact}}}{P_{\text{exact}}} \right|, \tag{35}$$

with $0.05(2^q-1)^{-1}$. In other words:

$$Er_2 \leq \frac{1}{2^q-1} 0.05. \tag{36}$$

Equation (36) can be considered as the basis of the accuracy-control addressed in [35, 92]. Consequently, to arrive at Equation (23), we can estimate Er_2, taking into account the step changes in Table 2, and substitute the estimation in Equation (36).

When the step in the second time integration computation is not half the step in the first computation (i.e. $\Delta t_2 \neq 0.5\Delta t_1$), from the definition of proper convergence or Figure 2 (considering Points 1 and 2 on the properly converging section of Figure 2; Point 1 on the right of Point 2):

$$\frac{\log(Er_1) - \log(Er_2)}{\log(\Delta t_1) - \log(\Delta t_2)} = q \qquad \text{for sufficiently small integration steps,} \tag{37}$$

which is equivalent to:

$$\frac{Er_2}{\Delta t_2{}^q} = \frac{Er_1}{\Delta t_1{}^q}. \tag{38}$$

From simple mathematics [55] and the fact that $\Delta t_2 \neq \Delta t_1$, Equation (38) leads to:

$$\frac{Er_2}{\Delta t_2{}^q} = \frac{Er_1}{\Delta t_1{}^q} = \frac{Er_1 - Er_2}{\Delta t_1{}^q - \Delta t_2{}^q}. \tag{39}$$

From Equation (39), and that close to the exact response, properly converging responses are either all more or all less than the exact response [139]:

$$\frac{Er_2}{\Delta t_2{}^q} = \frac{\left| \frac{P_1 - P_2}{P_{\text{exact}}} \right|}{\Delta t_1{}^q - \Delta t_2{}^q}. \tag{40}$$

Substituting the indices 1 and 2 in Equation (40), with $j-1$ and j, respectively, and using Equation (22), results in:

$$Er_j = \frac{\left| \frac{P_{j-1} - P_j}{P_{\text{exact}}} \right|}{\left(\frac{\Delta t_{j-1}}{\Delta t_j} \right)^q - 1} = \frac{\left| \frac{P_{j-1} - P_j}{P_{\text{exact}}} \right|}{\left(\frac{4n_{j-1}}{n_{j-1}+1} \right)^q - 1}. \tag{41}$$

Repeating the replacement of indices for Equation (36) leads to:

$$Er_j \leq \frac{1}{2^q - 1} 0.05. \tag{42}$$

Substituting Equation (41) in Equation (42) leads to:

$$\frac{\left| \dfrac{P_{j-1} - P_j}{P_{\text{exact}}} \right|}{\left(\dfrac{4n_{j-1}}{n_{j-1} + 1} \right)^q - 1} \leq \frac{0.05}{2^q - 1}. \tag{43}$$

Since n_{j-1} is larger than one and q is not smaller than one, Equation (43) can be re-written as:

$$\left| \frac{P_{j-1} - P_j}{P_{\text{exact}}} \right| \leq \frac{0.05}{2^q - 1} \left[\left(\frac{4n_{j-1}}{n_{j-1} + 1} \right)^q - 1 \right]. \tag{44}$$

In practice, P_{exact} is unavailable, and its closest substitute when convergence occurs properly is P_j (implied in Figure 2). Meanwhile, the order of accuracy is recommended not to be less than two and is generally equal to two [2, 23, 91]. Accordingly, Equation (44) can change form to:

$$\left| \frac{P_{j-1} - P_j}{P_j} \right| \leq \frac{0.05}{3} \left[\left(\frac{4n_{j-1}}{n_{j-1} + 1} \right)^2 - 1 \right], \tag{45}$$

which is identical to Equation (23) (the same result can be obtained, by concentrating on the control of Er_1 instead of Er_2). This makes the proof complete. Additionally, after each two sequential time integration computations, the following relation:

$$Er_j = 3 \left| \frac{P_{j-1} - P_j}{P_j} \right| \left[\left(\frac{4n_{j-1}}{n_{j-1} + 1} \right)^2 - 1 \right]^{-1}, \tag{46}$$

presents a measure for the accuracy of the final target response, not only for the ordinary time history analyses according to NZS 1170.5:2004 [35, 92], but also when the SEB THAAT is applied to the time history analyses according to the proposed procedure; for the former $n_j = n_{j-1} = 1$ (given Figure 2, the "3" in the right hand side seems to have a safety factor role).