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CONSTRUCTION OF ANGULAR-DEPENDENT POTENTIALS FROM TRIGONOMETRIC PÖSCHL-TELLER SYSTEMS WITHIN THE DUNKL FORMALISM

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Abstract. We generate solvable cases of the two angular equations resulting from variable separation in the three-dimensional Dunkl-Schrödinger equation expressed in spherical coordinates. It is shown that the Dunkl formalism interrelates these angular equations with trigonometric Pöschl-Teller systems. Based on this interrelation, we use point transformations and Darboux-Crum transformations to construct new solvable cases of the angular equations. Instead of the stationary energy, we use the constants due to the separation of variables as transformation parameters for our Darboux-Crum transformations.

Keywords: Dunkl operator, Schrödinger equation, trigonometric Pöschl-Teller potential, angular equation, Darboux-Crum transformation.

1. Introduction

Dunkl operators are commuting differential-difference operators associated with reflection groups [1] that play an important role in various areas of Mathematics and the natural sciences. For example, these operators are closely related to integrable systems of Calogero-Moser-Sutherland type [2–4], they are applied in the construction of angular momentum algebras [5, 6], for generalised Fourier analysis related to root systems [7], in nonlinear models [8], and in the generalisation of classical orthogonal polynomials through reflection groups [9]. A further field of application for Dunkl operators that has been studied extensively in recent years, is the construction of deformed quantum theories. Such deformations are obtained by replacing the conventional derivatives, as they appear for example in quantum momentum operators, through Dunkl operators [10]. By means of this process, we can construct a large variety of relativistic and nonrelativistic quantum systems within the Dunkl context. Particular cases that have recently been studied include the harmonic oscillator in the two-dimensional plane [11, 12], the planar Coulomb system [13], the Klein-Gordon equation for several interactions [14], rational extensions of the Dunkl oscillator [15], the relativistic harmonic oscillator [16, 17], and three-dimensional systems that allow for separation of variables in the Schrödinger equation [18]. In the present work, we focus on the latter type of systems. More precisely, we consider Dunkl-Schrödinger equations that allow for separation of variables in spherical coordinates. While the radial equation maintains its form within the Dunkl formalism, the two angular equations are different from their conventional counterparts. In particular, both can be taken to a form that resembles the conventional Schrödinger equation for a generalised trigonometric Pöschl-Teller potential [19]. Therefore, existing results and methods related to such potentials can be used directly for constructing new solvable cases of both angular equations, leading to an overall three-dimensional potential. While solutions for a particular class of potentials were obtained previously [20], here, our focus is more general. The purpose of the present work is to construct new solutions to the angular equations by incorporating existing results, and by adapting and applying the Darboux-Crum transformation to our equations. The article is organized as follows. In Section 2, we state the known general solution of the Schrödinger equation with trigonometric Pöschl-Teller potential. Since this equation have the same form as the angular equations resulting from the variable separation in the three-dimensional scenario, the solution presented in Section 2 can be adapted to those angular equations in a straightforward way. Besides this topic, Section 2 also briefly reviews the Darboux-Crum transformation that will be applied to the above mentioned angular equations. In Section 3, we begin to develop the method used in this work by defining the Hamiltonian within the Dunkl formalism and by generating the associated Schrödinger equation in spherical coordinates. The separation of variables then results, as usual, in a radial equation and two angular equations. In Section 4, one of these angular equations (the polar equation) is shown to match the form of the equation that was reviewed in Section 2. By means of this interrelation, solutions of the polar equation is constructed. These solutions are subsequently generalised by means of Darboux-Crum transformations, as introduced in Section 2. Section 5 essentially repeats the process from Section 4, but this time applied to the remaining angular equation (the azimuthal equation).

2. Preliminaries

In order to make this article self-contained, we will now summarise the results on the trigonometric version of the Pöschl-Teller potential, and on the closed-form solution of the associated Schrödinger equation. Furthermore, we briefly review the two algorithms of the Darboux-Crum transformation. Details on these topics can be found in [19, 21], and references therein.

2.1. The trigonometric Pöschl-Teller system

The stationary Schrödinger equation for the trigonometric Pöschl-Teller potential can be written in the form:

$$\Psi''(x) + \left[A + \frac{B}{\sin(x)^2} + \frac{C}{\cos(x)^2}\right] \Psi(x) = 0\,, \tag{1}$$

introducing real-valued constants A, B, and C. The general solution of (1) is given by:

$$\begin{aligned}\Psi(x) = {} & c_1 \; \sin(x)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-4B}} \; \cos(x)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}} \\ & \times {}_2F_1\left[\frac{1}{2} - \frac{\sqrt{A}}{2} + K_-, \frac{1}{2} + \frac{\sqrt{A}}{2} + K_-, 1 - \frac{\sqrt{1-4B}}{2}, \sin(x)^2\right] \\ & + c_2 \sin(x)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}} \cos(x)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}} \\ & \times {}_2F_1\left[\frac{1}{2} - \frac{\sqrt{A}}{2} + K_+, \frac{1}{2} + \frac{\sqrt{A}}{2} + K_+, 1 + \frac{\sqrt{1-4B}}{2}, \sin(x)^2\right],\end{aligned} \tag{2}$$

where c_1, c_2 denote the arbitrary linear factors, ${}_2F_1$ represents the hypergeometric function [22], and the abbreviation K that stands for:

$$K_\pm = \frac{\pm\sqrt{1-4B} + \sqrt{1-4C}}{4}\,. \tag{3}$$

Note that the parameters entering in the general Solution (2) are subject to restrictions, depending on the domain of Equation (1), and on the boundary conditions imposed on it.

2.2. The Darboux-Crum transformation

The purpose of the Darboux-Crum transformation is to provide an interrelation between the solutions of the following partner equations:

$$\Psi''(x) + \Big[E - U(x)\Big] \Psi(x) = 0\,, \tag{4}$$

$$\hat{\Psi}''(x) + \Big[E - \hat{U}(x)\Big] \hat{\Psi}(x) = 0\,. \tag{5}$$

Here, E denotes the stationary energy, and U, $\hat{U}$ stand for the respective potentials. These potentials, as well as the corresponding solutions Ψ and $\hat{\Psi}$ of the two equations, can be linked by the Darboux-Crum transformation, where we need to distinguish two algorithms.

- **Standard algorithm:** Let the transformation functions $\Psi_1, \ldots, \Psi_n$ be solutions of:

  $$\Psi_j''(x) + \Big[E_j - U(x)\Big] \Psi_j(x) = 0, \; j = 1, 2, \ldots, n\,,$$

  where the constants $E_1, \ldots, E_n$ are usually referred to as factorization energies or transformation energies. These names stem from the fact that for Schrödinger equations, these constants formally represent energies. However, in the present work, we will apply the Darboux-Crum transformation to equations that have the same form as (4), but are not Schrödinger equations, so that the constants $E_1, \ldots, E_n$ do not represent energies. For this reason we will refer to them as transformation parameters throughout this work. Further details will be discussed below at the beginning of Section 4.2. Now, a solution of equation (5) is given by:

  $$\hat{\Psi}(x) = \frac{W_{\Psi_1,\Psi_2,\ldots,\Psi_n,\Psi}(x)}{W_{\Psi_1,\Psi_2,\ldots,\Psi_n}(x)}\,, \tag{6}$$

  provided the potential $\hat{U}$ is constrained with its counterpart U as:

  $$\hat{U}(x) = U(x) - 2\frac{d^2}{dx^2} \log\left[W_{\Psi_1,\Psi_2,\ldots,\Psi_n}(x)\right]\,. \tag{7}$$

  Note that W in (6) and (7) denotes the Wronskian with respect to the functions in its index.

- **Confluent algorithm:** Let the transformation functions $\Psi_1, \ldots, \Psi_n$ be the solutions of the system:

$$\Psi_1''(x) + \Big[E_1 - U(x)\Big]\Psi_1(x) = 0\,, \tag{8}$$

$$\Psi_j''(x) + \Big[E_1 - U(x)\Big]\Psi_j(x) = -\Psi_{j-1}(x),\ j = 2, \ldots, n\,, \tag{9}$$

observe that there is only a single transformation parameter E_1. For example, in the case $n = 2$ (second-order transformation), the above system of equations reads:

$$\Psi_1''(x) + \Big[E_1 - U(x)\Big]\Psi_1(x) = 0\,,$$
$$\Psi_2''(x) + \Big[E_1 - U(x)\Big]\Psi_2(x) = -\Psi_1(x)\,.$$

Returning to the general case, if solutions $\Psi_1, \ldots, \Psi_n$ of the System (8), (9) are known, then the function:

$$\hat{\Psi}(x) = \frac{W_{\Psi_1,\Psi_2,\ldots,\Psi_n,\Psi}(x)}{W_{\Psi_1,\Psi_2,\ldots,\Psi_n}(x)}\,, \tag{10}$$

solves Equation (5), as long as the potential $\hat{U}$ is interrelated to its counterpart U by means of:

$$\hat{U}(x) = U(x) - 2\frac{d^2}{dx^2}\log\left[W_{\Psi_1,\Psi_2,\ldots,\Psi_n}(x)\right]\,. \tag{11}$$

Note that the Forms (10) and (11) are the same as (6) and (7), respectively, so that the only difference between the two algorithms lies in the equations that determine the transformation functions.

3. The Dunkl-Schrödinger system

Our first goal is to derive the stationary Schrödinger equation in three dimensions within the Dunkl formalism. This procedure is well known – see e.g. [18] – so we will not give the details of the derivation here. While the Hamiltonian governing our system is written in the usual form:

$$H = p_1^2 + \hat{p}_2^2 + \hat{p}_3^2 + V(x_1, x_2, x_3)\,, \tag{12}$$

where V represents the potential, and the momentum operators p_1, p_2, p_3 obey a nonstandard definition. Here, they are given as:

$$p_j = -iD_j = -i\left(\frac{\partial}{\partial x_j} + \frac{\nu_j}{x_j} - \frac{\nu_j}{x_j}R_j\right),\ j = 1, 2, 3\,, \tag{13}$$

introducing real-valued constants ν_1, ν_2, ν_3, and Dunkl operators D_1, D_2, D_3, that generalise the conventional partial derivative. Furthermore, the R_j stand for parity or reflection operators with respect to the variable x_j. This operator acts on the admissible functions Ψ in the following way:

$$R_1\ \Psi(x_1, x_2, x_3) = \Psi(-x_1, x_2, x_3)\,, \tag{14}$$

$$R_2\ \Psi(x_1, x_2, x_3) = \Psi(x_1, -x_2, x_3)\,, \tag{15}$$

$$R_3\ \Psi(x_1, x_2, x_3) = \Psi(x_1, x_2, -x_3)\,. \tag{16}$$

Note that setting $\nu_1 = \nu_2 = \nu_3 = 0$ converts the momentum operators in (13) to their conventional form, thus removing the reflection operators. Now, our Hamiltonian (12) is defined on a weighted Hilbert space L_w^2 with weight function:

$$w(x_1, x_2, x_3) = |x_1|^{2\nu_1}\ |x_2|^{2\nu_2}\ |x_3|^{2\nu_3}\,,$$

so that the norm of a function $\Psi \in L_w^2\left(\mathbb{R}^3\right)$ takes the form:

$$\|\Psi\| = \sqrt{\int\limits_{\mathbb{R}^3} \Psi(x_1, x_2, x_3)^* \Psi(x_1, x_2, x_3)|x_1|^{2\nu_1}|x_2|^{2\nu_2}|x_3|^{2\nu_3}\, dx_1\, dx_2\, dx_3}\,, \tag{17}$$

where the asterisk denotes a complex conjugation. It is understood that the admissible values of the parameters ν_1, ν_2, and ν_3 must be suitably restricted so that they do not contribute singularities in the integrand. In the next step we introduce spherical coordinates $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$ by means of the usual relations:

$$\begin{aligned} x_1 &= r\ \sin(\theta)\ \cos(\phi)\,, \\ x_2 &= r\ \sin(\theta)\ \sin(\phi)\,, \\ x_3 &= r\ \cos(\theta)\,. \end{aligned} \tag{18}$$

Furthermore, we restrict the class of potentials in (12) to functions of the following form:

$$V(r,\phi,\theta) = V_r(r) + \frac{1}{r^2} V_\theta(\theta) + \frac{1}{r^2 \sin(\theta)^2} V_\phi(\phi)\,, \tag{19}$$

introducing the three single-variable parameters V_r, V_θ, and V_ϕ. The purpose of representing our potential in the Form (19) is to ensure that we can separate variables in the Schrödinger equation that we will now construct. Writing our Hamiltonian (12) into the form of spherical Coordinates (18), and substituting the Potential (19), we obtain the Schrödinger equation $H\Psi = E\Psi$ in the form:

$$\begin{aligned} 0 = \Bigg\{ & \frac{\partial^2}{\partial r^2} + \frac{2}{r}\left(1+\nu_1+\nu_2+\nu_3\right)\frac{\partial}{\partial r} + E - V_r(r) \\ & + \frac{1}{r^2\sin(\theta)^2}\left\{\frac{\partial^2}{\partial \phi^2} - 2\Big[\nu_1\tan(\phi) - \nu_2\cot(\phi)\Big]\frac{\partial}{\partial\phi} - \frac{\nu_1(1-R_1)}{\cos(\phi)^2} - \frac{\nu_2(1-R_2)}{\sin(\phi)^2} - V_\phi(\phi)\right\} \\ & + \frac{1}{r^2}\left\{\frac{\partial^2}{\partial\theta^2} - 2\left[\nu_3\tan(\theta) - \left(\frac{1}{2}+\nu_1+\nu_2\right)\cot(\theta)\right]\frac{\partial}{\partial\theta} - \frac{\nu_3(1-R_3)}{\cos(\theta)^2} - V_\theta(\theta)\right\}\Bigg\}\,\Psi(r,\theta,\phi)\,. \end{aligned} \tag{20}$$

Recall that we can recover the known, conventional Schrödinger equation by removing the Dunkl context through the setting $\nu_1 = \nu_2 = \nu_3 = 0$. Next, in order to separate variables in (20), we first need to apply the reflection operators to the solution. By rewriting (14)–(16) to the case of spherical coordinates, we obtain:

$$R_1\ \Psi(r,\theta,\phi) = \Psi(r,\theta,\pi-\phi)\,, \tag{21}$$

$$R_2\ \Psi(r,\theta,\phi) = \Psi(r,\theta,-\phi)\,, \tag{22}$$

$$R_3\ \Psi(r,\theta,\phi) = \Psi(r,\pi-\theta,\phi)\,. \tag{23}$$

Without further information on the solution Ψ of Equation (20), application of the Actions (21)–(23) would render our equation in a form that does not allow for separation of variables. Therefore, we impose the following restrictions on Ψ:

$$R_j\ \Psi(r,\theta,\phi) = r_j\ \Psi(r,\theta,\phi),\ j=1,2,3\,, \tag{24}$$

introducing constants r_1, r_2, r_3, the existence and values of which must be determined separately for each particular potential (19). We refer to the r_1, r_2, r_3 as parity constants because they typically dictate that the function Ψ must exhibit a certain type of symmetry or parity. Now, after substituting (24) in (20), and requiring the usual factored form of the solution:

$$\Psi(r,\theta,\phi) = R(r)\ \Theta(\theta)\ \Phi(\phi)\,,$$

we can separate variables, so that (20) decomposes into three equations. The radial equation is given by:

$$R''(r) + \frac{2}{r}\left(1+\nu_1+\nu_2+\nu_3\right)\ R'(r) + \left[E - V_r(r) - \frac{\lambda}{r^2}\right]\ R(r) = 0\,, \tag{25}$$

where λ is the separation constant. Let us simplify the form of this equation by removing the first-order derivative term $\sim \Psi'$. To this end, we set:

$$R(r) = \frac{1}{r^{\nu_1+\nu_2+\nu_3}}\ S(r)\,,$$

which, after substitution in (25), gives our radial equation in the form:

$$S''(r) + \left[E - \frac{\lambda}{r^2} + \frac{(\nu_1+\nu_2+\nu_3)(\nu_1+\nu_2+\nu_3-1)}{r^2} - V_r(r)\right]\ S(r) = 0\,. \tag{26}$$

We observe that the terms containing the parameters ν_1, ν_2, and ν_3, can be formally the separation constant λ. As such, we can conclude that the general form of our radial equation is the same with and without the Dunkl context. For this reason, we will not consider this equation any further in the present work, except when stating examples. Next, the polar equation resulting from the separation of (20) takes the form:

$$\begin{aligned} & \Theta''(\theta) - 2\left[\nu_3\tan(\theta) - \left(\frac{1}{2}+\nu_1+\nu_2\right)\cot(\theta)\right]\Theta'(\theta) \\ & \qquad + \left[\lambda - \frac{m^2}{\sin(\theta)^2} - \frac{\nu_3(1-r_3)}{\cos(\theta)^2} - V_\theta(\theta)\right]\ \Theta(\theta) = 0\,, \end{aligned} \tag{27}$$

introducing another separation constant m^2. Finally, the azimuthal equation can be written as:

$$\Phi''(\phi) - 2\big[\nu_1 \tan(\phi) - \nu_2 \cot(\phi)\big]\ \Phi'(\phi) \\ + \left[m^2 - \frac{\nu_1(1-r_1)}{\cos(\phi)^2} - \frac{\nu_2(1-r_2)}{\sin(\phi)^2} - V_\phi(\phi)\right] \Phi(\phi) = 0\,. \tag{28}$$

We point out that the three Equations (25), (27) and (28) turn into their known counterparts if we remove the Dunkl context by setting $\nu_1 = \nu_2 = \nu_3 = 0$.

4. The polar equation

In this section we show that Equation (27) is closely related to trigonometric Pöschl-Teller systems. This relation can be used directly to find solutions of the polar equation for different potentials V_θ. Furthermore, these solutions can be generalised by the application of the Darboux-Crum transformation.

4.1. Extended trigonometric Pöschl-Teller systems

In order to observe the similarity between Equation (27) and trigonometric Pöschl-Teller systems, we rewrite the solution Θ in terms of a new function η as follows:

$$\Theta(\theta) = \cos(\theta)^{-\nu_3} \sin(\theta)^{-\nu_1-\nu_2-\frac{1}{2}} \eta(\theta)\,. \tag{29}$$

Substitution renders the polar Equation (27) in the form:

$$\eta''(\theta) + \left[\lambda + \frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} + \frac{(r_3-\nu_3)\nu_3}{\cos(\theta)^2} \right. \\ \left. + \frac{1-4m^2-4(\nu_1+\nu_2)^2}{4\sin(\theta)^2} - V_\theta(\theta)\right] \eta(\theta) = 0\,. \tag{30}$$

Comparison with (1) shows that this equation has Schrödinger form for a trigonometric Pöschl-Teller potential, extended by the function V_θ. As such, any extension that maintains solvability of (30), leads to a new polar term in the Potential (19), along with a solution of the associated Schrödinger Equation (20). Straightforward examples for such extensions V_θ are given by functions that resemble the shape of the trigonometric Pöschl-Teller potential, one of its special cases (as will be shown below), or their generalisations [23]. While the last case is beyond the scope of this article, let us consider a function V_θ that has the shape of a trigonometric Pöschl-Teller potential. We let:

$$V_\theta(\theta) = \alpha + \frac{\beta}{\sin(\theta)^2} + \frac{\gamma}{\cos(\theta)^2}\,, \tag{31}$$

where α, β and γ are constants. Observe that there are equivalent ways of expressing this function, for example, through tangent and cotangent functions. We have, for example, the two following representations:

$$V_\theta(\theta) = \alpha + \beta + \gamma + \gamma \tan(\theta)^2 + \frac{\beta}{\tan(\theta)^2}\,,$$
$$V_\theta(\theta) = \frac{\beta + (\alpha-\beta+\gamma)\sin(\theta)^2 - \alpha\sin(\theta)^4}{\sin(\theta)^2\cos(\theta)^2}\,,$$

both of which coincide with (31). By substituting (31) into (30), we obtain:

$$\eta''(\theta) + \left[\lambda + \alpha + \frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} \right. \\ \left. + \frac{(r_3-\nu_3)\nu_3+\gamma}{\cos(\theta)^2} + \frac{1-4m^2-4(\nu_1+\nu_2)^2+\beta}{4\sin(\theta)^2}\right] \eta(\theta) = 0\,. \tag{32}$$

This equation is exactly-solvable, as we can see from the comparison with (1). In addition, renaming the solution and the variable, we can match the two equations by setting:

$$A = \lambda + \alpha + \frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4}\,, \tag{33}$$
$$B = \frac{1}{4} - m^2 - (\nu_1+\nu_2)^2 + \frac{\beta}{4}\,, \tag{34}$$
$$C = (r_3-\nu_3)\nu_3 - \gamma\,. \tag{35}$$

Hence, implementing these settings in (2) provides the general solution of (32). Let us now continue by presenting an example that focuses on the special case of V_θ being a trigonometric Pöschl-Teller I potential. We set:

$$\alpha = \beta = 0\,. \tag{36}$$

In addition, we will aim for constructing a solution to our polar Equation (27) that forms part of a bound state to (20). Keeping this in mind, the inspection of the general Solution (2) shows that we must impose the condition $c_1 = 0$ in order to avoid singularities generated by the coefficient of the hypergeometric function. Furthermore, the hypergeometric function itself produces a singularity at $\theta = \pi/2$, unless it degenerates to a polynomial. It is known that this case occurs if its first argument equals to a nonpositive integer. From (2) and (3), we have the condition:

$$\frac{1}{2} - \frac{\sqrt{A}}{2} + \frac{\sqrt{1-4B}+\sqrt{1-4C}}{4} = -N,\ N = 0,1,2,\dots\,. \tag{37}$$

By substituting (33)–(35), and solving for the separation constant λ, we obtain:

$$\lambda = -\frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} + \frac{1}{4}\left[4N+2+2\sqrt{m^2+(\nu_1+\nu_2)^2}+\sqrt{1+4\gamma+4\nu_3(\nu_3-r_3)}\right]^2. \tag{38}$$

Now, taking into account the present Settings (33)–(38), we can get the solution of Equation (32) from (2) and (3) in the form:

$$\eta(\theta) = \sin(\theta)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}}\cos(\theta)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}}\,{}_2F_1\left[-N, \frac{1}{2}+\frac{\sqrt{A}}{2}+K_+, 1+\frac{\sqrt{1-4B}}{2}, \sin(\theta)^2\right], \tag{39}$$

where we have not substituted the explicit form of our parameters, as the resulting expression would become very long. In the next step, we build a solution Θ of our polar Equation (28) for the current settings by means of the point Transformation (29). This gives our Solution (39) as:

$$\Theta(\theta) = \sin(\theta)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}-\nu_1-\nu_2-\frac{1}{2}}\cos(\theta)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}-\nu_3} \times {}_2F_1\left[-N, \frac{1}{2}+\frac{\sqrt{A}}{2}+K_+, 1+\frac{\sqrt{1-4B}}{2}, \sin(\theta)^2\right]. \tag{40}$$

The final task consists in determining the constant r_3 that is inserted in (35). Recall that this constant results from applying the reflection operator R_3, as shown in (24). We can determine r_3 by evaluating (23), taking into account that it only acts on the function Θ. Since we know that:

$$R_3\ \sin(\theta) = \sin(\pi-\theta) = \sin(\theta)\,,$$
$$R_3\ \cos(\theta) = \cos(\pi-\theta) = -\cos(\theta)\,,$$

from (40) and (35) we obtain that:

$$R_3\Theta(\theta) = (-1)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}-\nu_3}\ \Theta(\theta) = (-1)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4(r_3-\nu_3)\nu_3+4\gamma}-\nu_3}\ \Theta(\theta)\,.$$

Comparison of this result with (24) gives the following equation for the constant r_3:

$$r_3 = (-1)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4(r_3-\nu_3)\nu_3+4\gamma}-\nu_3}\,. \tag{41}$$

Since we assume that r_3 is real-valued, the right side of this condition can only attain the two values -1 and 1, thus making it the only two possible values for r_3. Consequently, in the case $r_3 = -1$, the exponent of the right side of (41) must be an odd number, giving the condition:

$$\frac{1}{2} + \frac{1}{2}\sqrt{1+4(1+\nu_3)\nu_3+4\,\gamma} - \nu_3 = 2k+1\,, \tag{42}$$

where k denotes any integer. Observe further that if (42) is satisfied, then the function Θ in (40) is odd with respect to the point $\theta = \pi/2$. In the other case $r_3 = 1$, the exponent must be an even number, so that we must have:

$$\frac{1}{2} + \frac{1}{2}\sqrt{1-4(1-\nu_3)\nu_3+4\gamma} - \nu_3 = 2k\,, \tag{43}$$

for any integer k. If this condition is satisfied, then the function Θ in (40) is even with respect to the point $\theta = \pi/2$. In summary, the latter function is a solution to our polar Equation (27) for the Potential (31) with the Settings (3) and (33)–(38), provided the parameters are chosen to comply with (42) or (43). Let us now present examples for solutions of our polar equation for the overall settings:

$$\nu_1 = \nu_2 = \nu_3 = \frac{1}{2};\ m = 5\,. \tag{44}$$

Odd-parity solutions are obtained by setting $r_3 = -1$ and $\gamma = 8$, note that these parameter values satisfy the Constraint (42) for $k = 1$. Substituting the latter values and (44) in combination with (33)–(36) into (40) gives:

$$\Theta(\theta) = \sin(\theta)^{-1+\sqrt{26}} \cos(\theta)^3 \ {}_2F_1\left[-N, N+4+\sqrt{26}, 1+\sqrt{26}, \sin(\theta)^2 \right]. \tag{45}$$

Graphs of this function for different values of N can be found in Figure 1. For the even case, we set $r_3 = 1$, $\gamma = 4$, and keep the remaining parameters at the same values they were assigned in the previous case. From (40), we obtain that:

$$\Theta(\theta) = \sin(\theta)^{-1+\sqrt{26}} \cos(\theta)^2 \ {}_2F_1\left[-N, N+3+\sqrt{26}, 1+\sqrt{26}, \sin(\theta)^2 \right]. \tag{46}$$

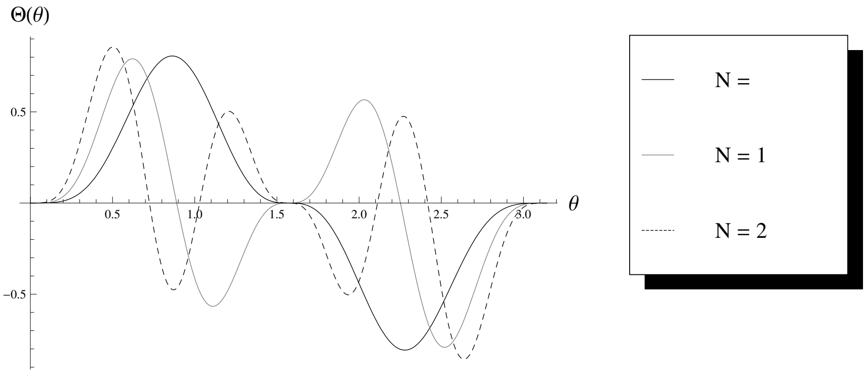

Figure 1. Graphs of odd Solutions (45) for different values of N.

Figure 2 shows graphs of this solution for several values of the parameter N.

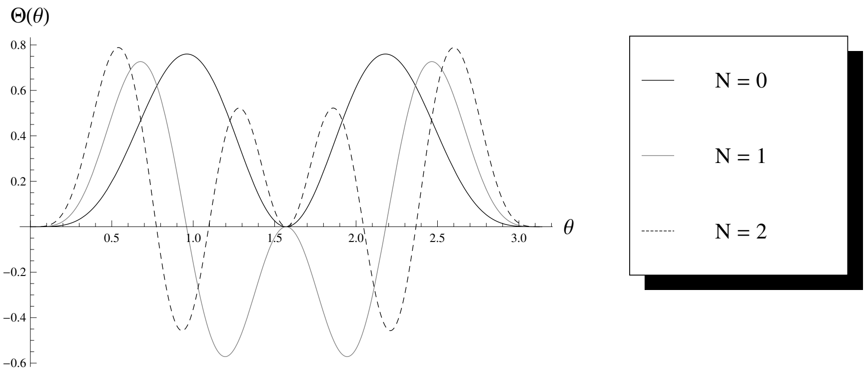

Figure 2. Graphs of even Solutions (46) for different values of N.

4.2. The Darboux-Crum transformation

In addition to using results on extended Pöschl-Teller systems directly in the polar Equation (27), we can also generate solvable cases of the latter equation by means of the Darboux-Crum transformation, if we consider our polar equation in the Form (30). As will be discussed below, this equation matches the form of (4), so that the Darboux-Crum transformation is applicable. It is important to stress that the transformation parameter, in our case, is not the stationary energy E of the system, as this energy does not enter in the polar equation.

Instead, we will use the separation constants λ and m^2 as our transformation parameters. More precisely, we will now show that each of these two separation constants can be used as a transformation parameter for a Darboux-Crum transformation, while the remaining parameters must be kept at fixed values. This is a strong restriction, as we will discuss further below. Before we continue, let us remark that in the vast majority of cases, where the Darboux-Crum transformation is applied to a Schrödinger equation, the transformation parameter is given by the stationary energy. This allows to control the discrete spectrum of the system, for instance, energy levels can be added or deleted. However, there are applications that involve non-energy transformation parameters. An example for such a case is given by the two-dimensional massless Dirac equation at zero energy, where the transformation parameter for the Darboux-Crum transformations is given by the wave number [24]. In another application to the one-dimensional Dunkl-Schrödinger equation, the transformation parameter consists of a constant involving the Dunkl constants [25]. As a result, transformed potentials in the latter reference are dependent on the stationary energy.

4.2.1. Transformation parameter λ

Our goal in this section is to apply our Darboux-Crum transformation to the polar equation, where the separation constant λ plays the role of the transformation parameter. Our starting point is Equation (30), where we observe that after renaming the solution and the variable, this equation matches (4), if we set:

$$\begin{aligned} E &= \lambda\,, \\ U(\theta) &= -\frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} - \frac{(r_3-\nu_3)\nu_3}{\cos(\theta)^2} - \frac{1-4m^2-4(\nu_1+\nu_2)^2}{4\sin(\theta)^2} + V_\theta(\theta)\,. \end{aligned} \tag{47}$$

Consequently, the Darboux-Crum transformation can be applied to Equation (30). Let us now assume that $\nu_1, \nu_2, \ldots, \nu_n$ are transformation functions that solve the following version of Equation (30):

$$\eta_j''(\theta) + \left[\lambda_j + \frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} + \frac{(r_3-\nu_3)\nu_3}{\cos(\theta)^2} \right. \\ \left. + \frac{1-4m^2-4(\nu_1+\nu_2)^2}{4\sin(\theta)^2} - V_\theta(\theta)\right] \eta_j(\theta) = 0,\ j = 1, \ldots, n\,, \tag{48}$$

where $\lambda_1, \ldots, \lambda_n$ are transformation parameters, and the potential V_θ is given by (31). Then, substituting into the Expressions (6) and (7) gives the results:

$$\hat{\eta}(\theta) = \frac{W_{\eta_1,\eta_2,\ldots,\eta_n,\eta}(\theta)}{W_{\eta_1,\eta_2,\ldots,\eta_n}(\theta)}\,, \tag{49}$$

$$\hat{V}_\theta(\theta) = V_\theta(\theta) - 2\frac{d^2}{d\theta^2}\log\left[W_{\eta_1,\eta_2,\ldots,\eta_n}(\theta)\right]. \tag{50}$$

Let us point out that due to (48), the function $\hat{\eta}$ depends on the transformation parameters $\lambda_1, \ldots, \lambda_n$, and also on the separation constant λ from Equation (30). Furthermore, the associated potential $\hat{V}_\theta$ also depends on the transformation parameters $\lambda_1, \ldots, \lambda_n$, but it does not depend on λ. The function $\hat{\eta}$ and the associated potential $\hat{V}_\theta$ are inserted in the transformed counterpart of (30) that reads:

$$\hat{\eta}''(\theta) + \left[\lambda + \frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} + \frac{(r_3-\nu_3)\nu_3}{\cos(\theta)^2} \right. \\ \left. + \frac{1-4m^2-4(\nu_1+\nu_2)^2}{4\sin(\theta)^2} - \hat{V}_\theta(\theta)\right] \hat{\eta}(\theta) = 0\,.$$

In the next step, we must rewrite this equation in the Form (27) that was obtained after the separation of variables. This form is given by:

$$\hat{\Theta}''(\theta) - 2\left[\nu_3\tan(\theta) - \left(\frac{1}{2}+\nu_1+\nu_2\right)\cot(\theta)\right]\hat{\Theta}'(\theta) \\ + \left[\lambda - \frac{m^2}{\sin(\theta)^2} - \frac{\nu_3(1-r_3)}{\cos(\theta)^2} - \hat{V}_\theta(\theta)\right]\hat{\Theta}(\theta) = 0\,, \tag{51}$$

where the functions $\hat{\Theta}$ and $\hat{V}_\theta$ will be determined now. In order to do so, we have to rewrite both (49) and (50) in terms of quantities pertaining to Equation (27). The functions $\eta_1, \eta_2, \ldots, \eta_n$ can expressed through solutions $\Theta_1, \Theta_2, \ldots, \Theta_n$ of the latter equation by means of the point Transformation (29). We obtain:

$$\Theta_j(\theta) = \cos(\theta)^{-\nu_3}\sin(\theta)^{-\nu_1-\nu_2-\frac{1}{2}}\eta_j(\theta),\ \ j = 1, 2, \ldots, n\,.$$

By using (29) once more for relating $\hat{\eta}$ to a new function $\hat{\Theta}$, we obtain from (49):

$$\begin{aligned}\hat{\Theta}(\theta) &= \cos(\theta)^{-\nu_3}\sin(\theta)^{-\nu_1-\nu_2-\frac{1}{2}}\ \hat{\eta}(\theta)\\ &= \cos(\theta)^{-\nu_3}\sin(\theta)^{-\nu_1-\nu_2-\frac{1}{2}}\ \frac{W_{\eta_1,\eta_2,\ldots,\eta_n,\eta}(\theta)}{W_{\eta_1,\eta_2,\ldots,\eta_n}(\theta)}\\ &= \frac{W_{\Theta_1,\Theta_2,\ldots,\Theta_n,\Theta}(\theta)}{W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)}\,.\end{aligned} \tag{52}$$

This is the explicit form of the transformed solution to our polar Equation (51). The associated potential can be obtained from (50) as follows:

$$\begin{aligned}\hat{V}_\theta(\theta) &= V_\theta(\theta) - 2\frac{d^2}{d\theta^2}\left\{\log\left[\cos(\theta)^{n\nu_3}\sin(\theta)^{n\left(\nu_1+\nu_2+\frac{1}{2}\right)}W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right]\right\}\\ &= V_\theta(\theta) - \frac{2n\left(\nu_1+\nu_2+\frac{1}{2}\right)}{\cos(\theta)^2} - \frac{2n\nu_3}{\sin(\theta)^2} - 2\frac{d^2}{d\theta^2}\log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right]\,.\end{aligned} \tag{53}$$

We observe that this transformed potential contributes terms of the trigonometric Pöschl-Teller form, independent of the initial potential V_θ. In summary, the Solution (52) satisfies the transformed polar Equation (51) with Potential (53). As indicated above, the latter potential, in general, depends on all parameters except on the transformation parameter λ. This includes the separation constant m^2 and the parity constant r_3. Hence, varying these parameters will also change the transformed potential. Since this is not desirable, the aforementioned parameters must be kept at a fixed value. Recall that this does not hold for λ.

Example: second-order Darboux-Crum transformation. Let us now give an example of a potential V_ϕ, for which the polar Equation (27) is in a solvable form, such that a Darboux-Crum transformation can be applied. We take the simplest case $V_\theta = 0$, that is, we have our initial equation in the form:

$$\begin{aligned}&\Theta''(\theta) - 2\left[\nu_3\tan(\theta) - \left(\frac{1}{2}+\nu_1+\nu_2\right)\cot(\theta)\right]\Theta'(\theta)\\ &\quad+\left[\lambda - \frac{m^2}{\sin(\theta)^2} - \frac{\nu_3(1-r_3)}{\cos(\theta)^2}\right]\Theta(\theta) = 0\,.\end{aligned}$$

We can obtain a particular solution of this equation from our prior calculations. More precisely, this solution is given by (40) for the settings:

$$\lambda = -\frac{(1+2\nu_1+2\nu_2+2\nu_3)^2}{4} + \left[2N+1+\sqrt{m^2+(\nu_1+\nu_2)^2}+\frac{1}{2}\sqrt{1+4\nu_3\ (\nu_3-r_3)}\right]^2, \tag{54}$$

$$A = \left[2N+1+\sqrt{m^2+(\nu_1+\nu_2)^2}+\frac{1}{2}\sqrt{1+4\nu_3(\nu_3-r_3)}\right]^2, \tag{55}$$

$$B = \frac{1}{4} - m^2 - (\nu_1+\nu_2)^2\,, \tag{56}$$

$$C = (r_3-\nu_3)\nu_3\,. \tag{57}$$

Now, in order to apply a Darboux-Crum transformation of second order, we must provide two transformation functions Θ_1 and Θ_2 that solve our initial equation for transformation parameters λ_1 and λ_2, given by (54), and determined by the values of N_1 and N_2. These values are chosen as:

$$N_1 = 1 \quad N_2 = 2\,.$$

By substituting them into (40) along with the Settings (54)–(57), we obtain our transformation functions Θ_1 and Θ_2. The first of these functions reads:

$$\begin{aligned}\Theta_1(\theta) &= \cos(\theta)^{\frac{1}{2}-\nu_3+\frac{1}{2}\sqrt{1+4\nu_3(\nu_3-r_3)}}\sin(\theta)^{-\nu_1-\nu_2+\sqrt{m^2+(\nu_1+\nu_2)^2}}\\ &\quad\times\left\{2+2\sqrt{m^2+(\nu_1+\nu_2)^2}-\sin(\theta)^2\left[4+2\sqrt{m^2+(\nu_1+\nu_2)^2}+\sqrt{1+4\nu_3(\nu_3-r_3)}\right]\right\},\end{aligned} \tag{58}$$

we omit the explicit form of the remaining transformation function Θ_2 due to its length. Let us now perform the Darboux-Crum transformation by inserting our transformation functions, and the Solution (40) with (54)–(57)

into (52) and (53). Since the resulting expressions are very long, we restrict ourselves here to show a particular case of the solution and the potential that enter in the transformed polar Equation (51). We choose our parameters as follows:

$$\nu_1 = 1 \quad \nu_2 = 2 \quad \nu_3 = 3 \quad m = 5\,. \tag{59}$$

Next, we insert them into (52). The evaluation gives the simplest solutions in the form:

$$\hat{\Theta}(\theta)_{|N=0,r_3=-1} = \frac{8(2483+426\sqrt{34})\cos(\theta)^3\sin(\theta)^{-1+\sqrt{34}}}{992+156\sqrt{34}-4(745+129\sqrt{34})\sin(\theta)^2+(2483+426\sqrt{34})\sin(\theta)^4}\,,$$

$$\hat{\Theta}(\theta)_{|N=0,r_3=1} = \frac{440(35+6\sqrt{34})\cos(\theta)^3\sin(\theta)^{-1+\sqrt{34}}}{896+148\sqrt{34}-4(637+109\sqrt{34})\sin(\theta)^2+55(35+6\sqrt{34})\sin(\theta)^4}\,.$$

Graphs of these functions can be found in Figures 3 and 4. Note that the parameter values chosen in the latter functions comply with the appropriate Conditions (42) and (43), respectively.

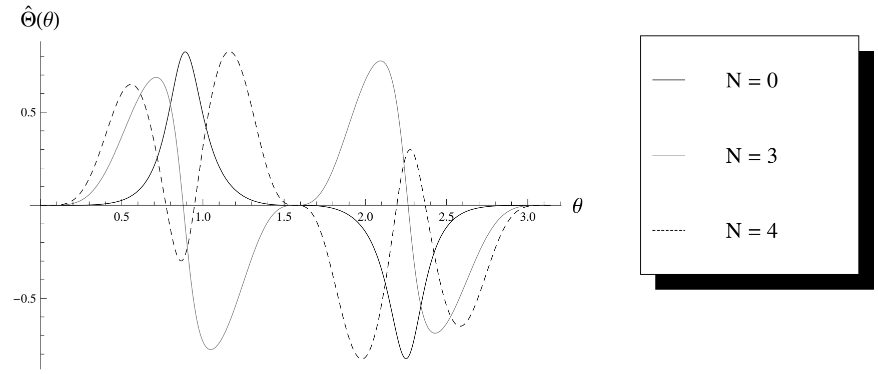

FIGURE 3. Graphs of odd Solutions (52) for the Settings (59), $r_3 = -1$, and different values of N. The transformation functions for the Darboux-Crum transformation are obtained from (40) with the Settings (54)–(57), and $N_1 = 1$, $N_2 = 2$.

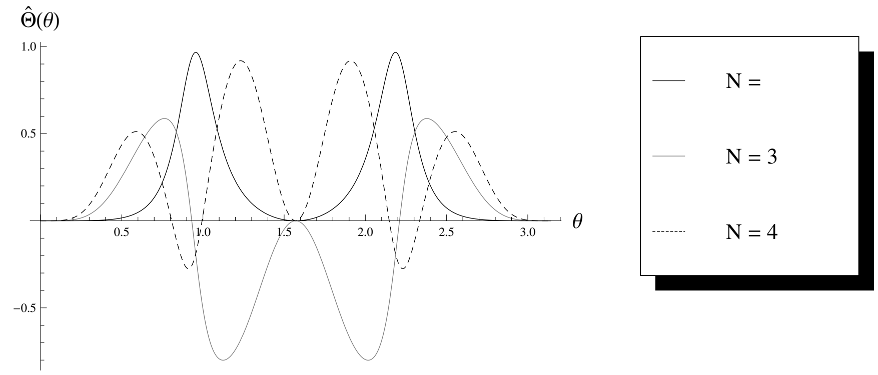

FIGURE 4. Graphs of even Solutions (52) for the Settings (59), $r_3 = 1$, and different values of N. The transformation functions for the Darboux-Crum transformation are obtained from (40) with the Settings (54)–(57), and $N_1 = 1$, $N_2 = 2$.

Before we conclude this example, let us state a particular case of the transformed potential $\hat{V}_\theta$ that is obtained from (53), as described above. Upon substituting the parameter Settings (59) and $r_3 = -1$, we find:

$$\begin{aligned}\hat{V}_\theta(\theta) = &\Bigg\{4\Big[-89\ (14609723 + 2505554\sqrt{34})\cos(2\theta) - 10(53136719 + 9112838\sqrt{34}) \\ &\times \cos(4\theta) - 29(1632977 + 280054\sqrt{34})\cos(6\theta) + \frac{2048(385457 + 66281\sqrt{34})}{\sin(\theta)^2} \\ &+ \frac{313632(361 + 60\sqrt{34})}{\cos(\theta)^2} - 398655716\sqrt{34} - 2327701978\Big]\Bigg\} \\ &\times \Bigg\{\Big[4(497 + 90\sqrt{34})\cos(2\theta) + (2483 + 426\sqrt{34})\cos(4\theta) + 231(15 + 2\sqrt{34}))^2\Big]\Bigg\}^{-1}.\end{aligned} \tag{60}$$

The graph of this potential, along with its counterpart for $r_3 = 1$, can be seen in Figure 5.

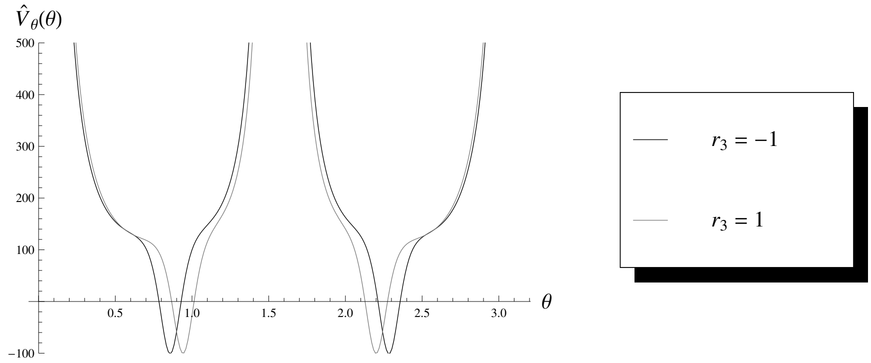

Figure 5. Graphs of the Potential (60) and its counterpart for the two admissible values of r_3.

4.2.2. Transformation parameter $\mathbf{m^2}$

Instead of λ, we can also use the separation constant m^2 as a Darboux-Crum transformation parameter. In order to do so, we must change the form of our equation, so that it matches (4). This requires a point transformation that changes both the dependent and the independent variable. We set:

$$\theta(y) = 2\text{arccot}[\exp(iy)]\,, \tag{61}$$

$$\eta(y) = \frac{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}}{\left[\exp(2iy)+1\right]^{\nu_1+\nu_2+\nu_3}}\ \Theta\{2\text{arccot}[\exp(iy)]\}\,, \tag{62}$$

introducing a new function η. By inserting this point transformation in (27), we obtain our equation as:

$$\begin{aligned}\eta''(y) + \Bigg\{&m^2 + (\nu_1+\nu_2)^2 - \frac{\lambda + (\nu_1+\nu_2+\nu_3)(1+\nu_1+\nu_2+\nu_3)}{\cos(y)^2} \\ &+ \frac{(r_3-\nu_3)\nu_3}{\sin(y)^2} + \frac{1}{\cos(y)^2}V_\theta\{2\text{arccot}[\exp(iy)]\}\Bigg\}\,\eta(y) = 0\,.\end{aligned} \tag{63}$$

Note that the coordinate Change (61) in our point transformation was necessary because in (27) the separation constant m^2 has a nonconstant coefficient that needs to be removed. We observe that Equations (63) and (4) can now be matched, if we choose:

$$\begin{aligned}E &= m^2\,, \\ U(y) &= -(\nu_1+\nu_2)^2 + \frac{\lambda + (\nu_1+\nu_2+\nu_3)(1+\nu_1+\nu_2+\nu_3)}{\cos(y)^2} \\ &\quad - \frac{(r_3-\nu_3)\nu_3}{\sin(y)^2} - \frac{1}{\cos(y)^2}V_\theta\{2\text{arccot}[\exp(iy)]\}\,.\end{aligned}$$

Consequently, the Darboux-Crum transformation can be applied to Equation (63). In order to apply this transformation, we need:

$$\eta_j''(y) + \left\{ m_j^2 + (\nu_1+\nu_2)^2 - \frac{\lambda + (\nu_1+\nu_2+\nu_3)(1+\nu_1+\nu_2+\nu_3)}{\cos(y)^2} \right.$$
$$\left. + \frac{(r_3-\nu_3)\nu_3}{\sin(y)^2} + \frac{1}{\cos(y)^2} V_\theta\{2\mathrm{arccot}[\exp(iy)]\} \right\} \eta_j(y) = 0\,, \tag{64}$$

where $j = 1, \ldots, n$, the transformation parameters are given by $m_1^2, \ldots, m_n^2$, and the potential V_θ is provided by (31). Now, evaluation of (6) and (7) gives:

$$\hat{\eta}(y) = \frac{W_{\eta_1,\eta_2,\ldots,\eta_n,\eta}(y)}{W_{\eta_1,\eta_2,\ldots,\eta_n}(y)}\,, \tag{65}$$

$$\hat{V}_\theta\{2\mathrm{arccot}[\exp(iy)]\} = V_\theta\{2\mathrm{arccot}[\exp(iy)]\} + 2\cos(y)^2 \frac{d^2}{dy^2} \log\left[W_{\eta_1,\eta_2,\ldots,\eta_n}(y)\right]. \tag{66}$$

Here, the function $\hat{\eta}$ depends on the transformation parameters $m_1^2, \ldots, m_n^2$, and also on the separation constant m^2 from Equation (63). Furthermore, the associated potential $\hat{V}_\theta$ also depends on the transformation parameters $m_1^2, \ldots, m_n^2$, but it does not depend on m^2. The function $\hat{\eta}$ and the potential $\hat{V}_\theta$ are inserted in the transformed version of (63), which reads:

$$\hat{\eta}''(y) + \left\{ m^2 + (\nu_1+\nu_2)^2 - \frac{\lambda + (\nu_1+\nu_2+\nu_3)(1+\nu_1+\nu_2+\nu_3)}{\cos(y)^2} \right.$$
$$\left. + \frac{(r_3-\nu_3)\nu_3}{\sin(y)^2} + \frac{1}{\cos(y)^2} \hat{V}_\theta\{2\mathrm{arccot}[\exp(iy)]\} \right\} \hat{\eta}(y) = 0\,.$$

Now we need to revert our point Transformation (61), (62), so that this equation is converted back to the Form (51). A solution $\hat{\Theta}$ can be found in two steps, the first of which is the implementation of (62). We obtain:

$$\begin{aligned}\hat{\Theta}\{2\mathrm{arccot}[\exp(iy)]\} &= \frac{[\exp(2iy)+1]^{\nu_1+\nu_2+\nu_3}}{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}}\ \hat{\eta}(y) \\ &= \frac{[\exp(2iy)+1]^{\nu_1+\nu_2+\nu_3}}{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}} \left[\frac{W_{\eta_1,\eta_2,\ldots,\eta_n,\eta}(y)}{W_{\eta_1,\eta_2,\ldots,\eta_n}(y)}\right].\end{aligned} \tag{67}$$

In the second step, we invert the coordinate Change (61). Taking into account that:

$$y(\theta) = -i\log\left[\cot\left(\frac{\theta}{2}\right)\right],$$

and defining functions $\Theta_1, \Theta_2, \ldots, \Theta_n$ by reverting (61), (62) as:

$$\Theta_j(\theta) = \frac{\tan\left(\frac{\theta}{2}\right)^{\nu_1+\nu_2}}{\cos(\theta)^{\nu_3}\sin\left(\frac{\theta}{2}\right)^{2\nu_1+2\nu_2}}\,\eta_j\left\{-i\log\left[\cot\left(\frac{\theta}{2}\right)\right]\right\},\ j = 1,2,\ldots,n\,,$$

we find, after substitution into (67) and simplification of the Wronskians, that:

$$\hat{\Theta}(\theta) = \sin(\theta)^n\,\frac{W_{\Theta_1,\Theta_2,\ldots,\Theta_n,\Theta}(\theta)}{W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)}\,. \tag{68}$$

The remaining task is to rewrite the transformed Potential (66) in terms of the functions $\Theta_1, \Theta_2, \ldots, \Theta_n$, and the coordinate θ. By implementing (61) and (62) in a first step:

$$\begin{aligned}\hat{V}_\theta(\theta) &= V_\theta(\theta) \\ &+ \frac{2}{\sin(\theta)^2}\left(\frac{d^2}{dy^2}\log\left\{\frac{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}}{[\exp(2iy)+1]^{\nu_1+\nu_2+\nu_3}}\,W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(y)\right\}\right)_{|y=y(\theta)}.\end{aligned} \tag{69}$$

In the next step, we focus on simplifying the second derivative term that we abbreviate as L. By expanding the logarithm, we find:

$$L(\theta) = \left(\frac{d^2}{dy^2} \log \left\{ \frac{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}}{\left[\exp(2iy)+1\right]^{\nu_1+\nu_2+\nu_3}} W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(y) \right\} \right)_{|y=y(\theta)}$$
$$= \left(\frac{d^2}{dy^2} \log \left\{ \frac{\exp\left[i(\nu_1+\nu_2)y\right]\left[\exp(2iy)-1\right]^{\nu_3}}{\left[\exp(2iy)+1\right]^{\nu_1+\nu_2+\nu_3}} \right\} + \frac{d^2}{dy^2} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(y)\right] \right)_{|y=y(\theta)} . \tag{70}$$

Now we can evaluate the second derivative in the first term, and change coordinates to θ. We obtain:

$$L(\theta) = \left[\frac{\nu_1+\nu_2+\nu_3}{\cos(y)^2} - \frac{\nu_3}{\sin(y)^2} \right]_{|y=y(\theta)} + \left\{ \frac{d^2}{dy^2} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(y)\right] \right\}_{|y=y(\theta)}$$
$$= \sin(\theta)^2 \left[\nu_1+\nu_2+\frac{\nu_3}{\cos(\theta)^2} \right] + \left\{ \frac{d^2}{dy^2} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(y)\right] \right\}_{|y=y(\theta)} .$$

We proceed by changing the coordinate in our remaining logarithmic term. After rewriting the second derivative in terms of the variable θ, and applying this derivative, we arrive at:

$$L(\theta) = \sin(\theta)^2 \left[\nu_1+\nu_2+\frac{\nu_3}{\cos(\theta)^2} \right] - \left[\sin(\theta)^2 \frac{d^2}{d\theta^2} + \frac{1}{2}\sin(2\,\theta)\,\frac{d}{d\theta} \right] \log\left\{ \left[-i\sin(\theta)\right]^{\frac{n(n-1)}{2}} W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta) \right\}$$
$$= \sin(\theta)^2 \left[\nu_1+\nu_2+\frac{\nu_3}{\cos(\theta)^2} \right] + \frac{n(n-1)}{2} + \sin(\theta)^2$$
$$\times \frac{d^2}{d\theta^2} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right] + \frac{n(1-n)}{2} \cos(\theta)^2 \frac{d}{d\theta} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right] .$$

By combining this result with (69), we obtain our transformed potential in the form:

$$\hat{V}_\theta(\theta) = V_\theta(\theta) + 2\left[\nu_1+\nu_2+\frac{\nu_3}{\cos(\theta)^2} \right] + \frac{n(n-1)}{\sin(\theta)^2} + 2\frac{d^2}{d\theta^2} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right]$$
$$+ n(1-n)\cot(\theta)^2 \frac{d}{d\theta} \log\left[W_{\Theta_1,\Theta_2,\ldots,\Theta_n}(\theta)\right] . \tag{71}$$

In conclusion, the Function (68) solves our transformed polar Equation (51) for the Potential (71). Recall that the latter potential generally depends on all parameters except on m^2. It is therefore, desirable to keep the latter parameters at fixed values. As a further comment, let us point out that the transformed potentials, for the two cases of λ and m^2 being the transformation parameters, are entirely different.

5. The azimuthal equation

Similar to the previous case, the azimuthal equation can be linked to trigonometric Pöschl-Teller systems. In addition to using results on those systems to construct solutions, we will show Darboux-Crum transformations to be applicable. In contrast to the polar equation, this time there is only one transformation parameter.

5.1. Extended trigonometric Pöschl-Teller systems

We will approach the azimuthal Equation (28) by first converting it in into a form similar to the conventional Schrödinger form. To do this, we use the point transformation:

$$\Phi(\phi) = \cos(\phi)^{-\nu_1} \sin(\phi)^{-\nu_2} \xi(\phi) , \tag{72}$$

which transforms (28) into:

$$\xi''(\phi) + \left[m^2 + (\nu_1+\nu_2)^2 + \frac{(r_1-\nu_1)\nu_1}{\cos(\phi)^2} + \frac{(r_2-\nu_2)\nu_2}{\sin(\phi)^2} - V_\phi(\phi) \right] \xi(\phi) = 0 . \tag{73}$$

Similar to the polar Equation (51), this can be interpreted as a Schrödinger equation for a generalisation of the trigonometric Pöschl-Teller potential. While the two equations are very similar in this sense, the main differences are in the domains and in the presence of two parity parameters in (73). To give a simple example of a solution to (73), we choose the function V_ϕ as:

$$V_\phi(\phi) = \frac{\gamma}{\cos(\phi)^2}\,, \tag{74}$$

introducing a real-valued constant γ. By substituting V_ϕ into (73), we obtain (73) in a form that matches (4), provided we choose the parameters A, B, and C as follows:

$$A = m^2 + (\nu_1 + \nu_2)^2\,, \tag{75}$$

$$B = (r_2 - \nu_2)\ \nu_2\,, \tag{76}$$

$$C = (r_1 - \nu_1)\ \nu_1 - \gamma\,. \tag{77}$$

Furthermore, we will employ the Condition (37) here, which requires that the separation constant m^2 is given by:

$$m^2 = -(\nu_1 + \nu_2)^2 + \left[1 + 2N + \frac{1}{2}\sqrt{1 + 4\gamma + 4\nu_1(\nu_1 - r_1)} + \frac{1}{2}\sqrt{1 + 4\nu_2(\nu_2 - r_2)}\right]^2. \tag{78}$$

Now, according to (2) and (3), a particular solution of (73) can be written in the form:

$$\xi(\phi) = \sin(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}}\cos(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}}\ {}_2F_1\left[-N, \frac{1}{2} + \frac{\sqrt{A}}{2} + K_+, 1 + \frac{\sqrt{1-4B}}{2}, \sin(\phi)^2\right], \tag{79}$$

recall that the parameters A, B, C are defined in (75)–(77), where A depends on m^2 from (78). In the next step, we use our point Transformation (72) to build the solution for our azimuthal Equation (28) with (74). The result reads:

$$\Phi(\phi) = \sin(\phi)^{-\nu_2+\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}}\cos(\phi)^{-\nu_1+\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}}\ {}_2F_1\left[-N, \frac{1}{2} + \frac{\sqrt{A}}{2} + K_+, 1 + \frac{\sqrt{1-4B}}{2}, \sin(\phi)^2\right]. \tag{80}$$

We now need to find the values of our parity parameters r_1 and r_2 using the Relations (21), (22), and (24). Before we do so, it is convenient to find the actions of the reflection operators on the sine and cosine functions. We have:

$$R_1\ \sin(\phi) = \sin(\phi)\,,$$
$$R_1\ \cos(\phi) = -\cos(\phi)\,,$$
$$R_2\ \sin(\phi) = -\sin(\phi)\,,$$
$$R_2\ \cos(\phi) = \cos(\phi)\,.$$

Thus, applying these operators to the Solution (80) results in:

$$R_1\ \Phi(\phi) = (-1)^{-\nu_1+\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}}\Phi(\phi)\,,$$
$$R_2\ \Phi(\phi) = (-1)^{-\nu_2+\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}}\Phi(\phi)\,.$$

Now, if we combine this with (21) and (22), we obtain conditions for the parity parameters r_1 and r_2. These read:

$$r_1 = (-1)^{-\nu_1+\frac{1}{2}+\frac{1}{2}\sqrt{1-4C}}\,, \tag{81}$$

$$r_2 = (-1)^{-\nu_2+\frac{1}{2}+\frac{1}{2}\sqrt{1-4B}}\,, \tag{82}$$

note that the constants B and C are defined in (76) and (77), respectively. The first of these conditions is satisfied if the exponent equals an even or an odd number, depending on the value of r_1. We have the two conditions:

$$-\nu_1 + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\gamma + 4\nu_1(\nu_1 - r_1)} = 2k_1 + 1\,, \tag{83}$$

$$-\nu_1 + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\gamma + 4\nu_1(\nu_1 - r_1)} = 2k_1\,, \tag{84}$$

where k_1 is an integer. If (83) is satisfied, then $r_1 = -1$, meaning that the Solution (80) has odd parity with respect to the points $\phi = \pi/2$ and $\phi = 3\pi/2$. In the other case that the parameters comply with (84), then $r_1 = 1$, so that (80) has even parity with respect to the points $\phi = \pi/2$ and $\phi = 3\pi/2$. A similar way of reasoning can be applied to the second Condition (82). We have:

$$-\nu_2 + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_2(\nu_2 - r_2)} = 2k_2 + 1\,, \tag{85}$$

$$-\nu_2 + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_2(\nu_2 - r_2)} = 2k_2\,, \tag{86}$$

for an integer k_2. If (85) is satisfied, then $r_2 = -1$, so that (80) is odd with respect to the origin. Otherwise, If (86) is true, then $r_2 = 1$, meaning that (80) is even with respect to the origin. Let us show examples of our Solutions (80) for the following parameter setting:

$$\nu_1 = 2 \quad \nu_2 = -\frac{1}{2} \quad \nu_3 = 2 \quad \gamma = 14\,. \tag{87}$$

Figures 6 and 7 show graphs of the Functions (80) for the parameter Setting (87), and different choices for the parity parameters r_1 and r_2.

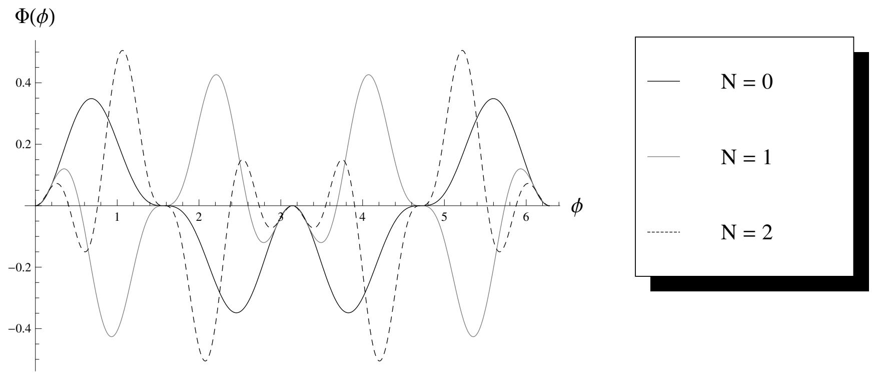

FIGURE 6. Graphs of the Solution (80) for the parameter Settings (87) and $r_1 = -1$, $r_2 = 1$ for different values of N.

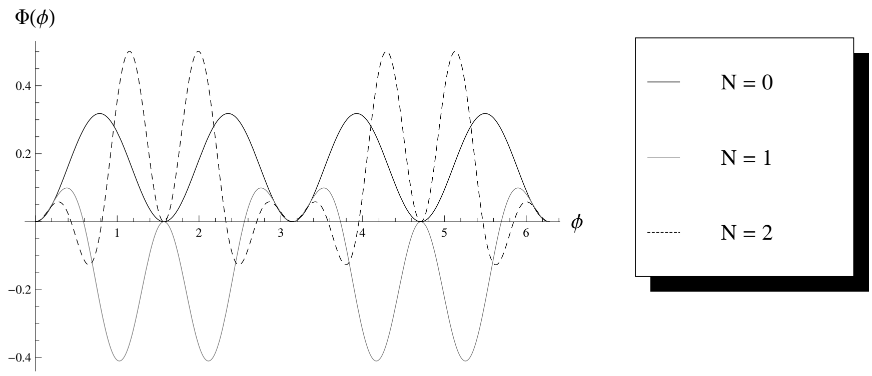

FIGURE 7. Graphs of the Solution (80) for the parameter Settings (87) and $r_1 = r_2 = 1$ for different values of N.

It is important to point out that the parameter settings used in Figures 6 and 7, are compatible with the Conditions (81) and (82).

5.2. The Darboux-Crum transformation

We will follow an analogous approach as in Section 4.1. There is only one separation constant m^2 that appears in our equation. This separation constant will act as transformation parameter in the Darboux-Crum transformation. As a first step we see that (73) matches (4) if the following settings are made:

$$E = m^2\,,$$

$$U(\phi) = -(\nu_1 + \nu_2)^2 - \frac{(r_1 - \nu_1)\nu_1}{\cos(\phi)^2} - \frac{(r_2 - \nu_2)\nu_2}{\sin(\phi)^2} + V_\phi(\phi)\,. \tag{88}$$

Since these settings make our Equations (4) and (73) match, our Darboux-Crum transformation becomes applicable. Before we construct this transformation, let us observe that (73) can be interpreted as a Schrödinger equation for an extended trigonometric Pöschl-Teller potential, similar to the rewritten polar Equations (30) and (63). For our Darboux-Crum transformation, we require transformation functions $\xi_1, \xi_2, \ldots, \xi_n$ that are solutions to the following version of Equation (73):

$$\xi_j''(\phi) + \left[m_j^2 + (\nu_1+\nu_2)^2 + \frac{(r_1-\nu_1)\nu_1}{\cos(\phi)^2} + \frac{(r_2-\nu_2)\nu_2}{\sin(\phi)^2} - V_\phi(\phi)\right] \xi_j(\phi) = 0,\ j = 1, \ldots, n\,. \tag{89}$$

Here, the constants $m_1^2, \ldots, m_n^2$ are the transformation parameters, and the potential V_ϕ is given by (74). Evaluation of (6) and (7) then gives:

$$\hat{\xi}(\phi) = \frac{W_{\xi_1,\xi_2,\ldots,\xi_n,\xi}(\phi)}{W_{\xi_1,\xi_2,\ldots,\xi_n}(\phi)}\,, \tag{90}$$

$$\hat{V}_\phi(\phi) = V_\phi(\phi) - 2\frac{d^2}{d\phi^2}\log\left[W_{\xi_1,\xi_2,\ldots,\xi_n}(\phi)\right]. \tag{91}$$

For the sake of clarity, let us point out that the function $\hat{\xi}$ is dependent on the transformation parameters $m_1^2, \ldots, m_n^2$, as well as on the separation constant m^2 from (73). The potential $\hat{V}_\phi$ is dependent on the transformation parameters $m_1^2, \ldots, m_n^2$, but not on m^2. Now, after applying the Darboux-Crum transformation, these two functions are inserted into the resulting equation as follows:

$$\hat{\xi}''(\phi) + \left[m^2 + (\nu_1+\nu_2)^2 + \frac{(r_1-\nu_1)\nu_1}{\cos(\phi)^2} + \frac{(r_2-\nu_2)\nu_2}{\sin(\phi)^2} - \hat{V}_\phi(\phi)\right]\hat{\xi}(\phi) = 0\,.$$

We now need to convert this equation into a transformed version of (28) that reads:

$$\begin{aligned}&\hat{\Phi}''(\phi) - 2\big[\nu_1\tan(\phi) - \nu_2\cot(\phi)\big]\ \Phi'(\phi)\\ &\qquad + \left[m^2 - \frac{\nu_1(1-r_1)}{\cos(\phi)^2} - \frac{\nu_2(1-r_2)}{\sin(\phi)^2} - \hat{V}_\phi(\phi)\right]\hat{\Phi}(\phi) = 0\,.\end{aligned} \tag{92}$$

The solution of this equation can be found from (90) by introducing functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ through our point Transformation (72) as:

$$\Phi_j(\phi) = \cos(\phi)^{-\nu_1}\sin(\phi)^{-\nu_2}\xi_j(\phi),\ j = 1, 2, \ldots, n\,. \tag{93}$$

Substitution into (90) gives, after simplification:

$$\hat{\Phi}(\phi) = \frac{W_{\Phi_1,\Phi_2,\ldots,\Phi_n,\Phi}(\phi)}{W_{\Phi_1,\Phi_2,\ldots,\Phi_n}(\phi)}\,. \tag{94}$$

The transformed potential $\hat{V}$ inserted into Equation (73) is obtained by combining (72) and (91). Substitution yields:

$$\begin{aligned}\hat{V}_\phi(\phi) &= V_\phi(\phi) - 2\frac{d^2}{d\phi^2}\log\left[\cos(\phi)^{n\nu_1}\sin(\phi)^{n\nu_2}W_{\Phi_1,\Phi_2,\ldots,\Phi_n}(\phi)\right]\\ &= V_\phi(\phi) + 2n\left[\frac{\nu_1}{\cos(\phi)^2} + \frac{\nu_2}{\sin(\phi)^2}\right] - 2\frac{d^2}{d\phi^2}\log\left[W_{\Phi_1,\Phi_2,\ldots,\Phi_n}(\phi)\right].\end{aligned} \tag{95}$$

We observe that this potential contributes a term of trigonometric Pöschl-Teller type, independent of its initial counterpart V_ϕ. Since (95) generally depends on all parameters in the Function (88), these parameters must be assigned fixed values in order not to change the potential. This is not valid for the transformation parameter m^2, since (95) does not depend on it. In summary, the transformed Equation (92) for this potential is solved by the Function (94).

Example: confluent second-order Darboux-Crum transformation. In this application, we will generate a solvable case of the azimuthal equation by means of a confluent Darboux-Crum transformation. Our starting point is Equation (28) for the potential $V_\phi = 0$. This results in equation:

$$\Phi''(\phi) - 2\left[\nu_1\tan(\phi) - \nu_2\cot(\phi)\right]\Phi'(\phi) + \left[m^2 - \frac{\nu_1(1-r_1)}{\cos(\phi)^2} - \frac{\nu_2(1-r_2)}{\sin(\phi)^2}\right]\Phi(\phi) = 0\,. \tag{96}$$

In order to perform a confluent Darboux-Crum transformation of second order, we resort to the version of our equation that is obtained after applying the point Transformation (72). This equation is given by (73) for $V_\phi = 0$, that is, we have:

$$\xi''(\phi) + \left[m^2 + (\nu_1+\nu_2)^2 + \frac{(r_1-\nu_1)\nu_1}{\cos(\phi)^2} + \frac{(r_2-\nu_2)\nu_2}{\sin(\phi)^2}\right]\xi(\phi) = 0\,.$$

A particular solution is found from (79), where the parameters must be chosen according to (75)–(78) for $\gamma = 0$. This gives:

$$\begin{aligned}
A &= \left[1 + 2N + \frac{1}{2}\sqrt{1+4\nu_1(\nu_1-r_1)} + \frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}\right]^2,\\
B &= (r_2-\nu_2)\nu_2\,,\\
C &= (r_1-\nu_1)\nu_1\,,\\
m^2 &= -(\nu_1+\nu_2)^2 + \left[1 + 2N + \frac{1}{2}\sqrt{1+4\nu_1(\nu_1-r_1)} + \frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}\right]^2.
\end{aligned} \tag{97}$$

In the next step we must provide two transformation functions ξ_1 and ξ_2, together with a single transformation parameter m_1^2. This parameter is determined by choosing a value for the constant N in (97). In the present case, we choose $N = N_1 = 0$, so we can extract the first transformation function from (79). This gives:

$$\xi_1(\phi) = \xi(\phi)_{|N_1=0} = \cos(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_1(\nu_1-r_1)}}\sin(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}}\,. \tag{98}$$

The remaining transformation function is a solution of Equation (9) for $j = 2$ and the present settings. This equation reads:

$$\xi_2''(\phi) + \left[m^2 + (\nu_1+\nu_2)^2 + \frac{(r_1-\nu_1)\nu_1}{\cos(\phi)^2} + \frac{(r_2-\nu_2)\nu_2}{\sin(\phi)^2}\right]\xi_2(\phi) = -\xi_1(\phi)\,, \tag{99}$$

recall that m^2 and ξ_1 are given in (97) and (98), respectively. A solution of (99) can be found by taking the derivative of ξ with respect to the transformation parameter [26]. Applying the chain rule gives:

$$\begin{aligned}
\xi_2(\phi) &= \left[\frac{d}{d(m^2)}\;\xi(\phi)\right]_{|N=N_1=0}\\
&= \left\{\left[4 + 8N + 2\sqrt{1+4\nu_1(\nu_1-r_1)} + 2\sqrt{1+4\nu_2(\nu_2-r_2)}\right]^{-1}\frac{d}{dN}\xi(\phi)\right\}_{|N=N_1=0}\\
&= \frac{1}{4 + 2\sqrt{1+4\nu_1(\nu_1-r_1)} + 2\sqrt{1+4\nu_2(\nu_2-r_2)}}\\
&\times \cos(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_1(\nu_1-r_1)}}\sin(\phi)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}}\\
&\times \left\{\frac{d}{dN}{}_2F_1\left[-N, 1+N+\frac{1}{2}\sqrt{1+4\nu_1(\nu_1-r_1)} + \frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}, 1\right.\right.\\
&\left.\left. + \frac{1}{2}\sqrt{1+4\nu_2(\nu_2-r_2)}, \sin(\phi)^2\right]\right\}_{|N=N_1=0}.
\end{aligned} \tag{100}$$

We substitute the transformation Functions (98) and (100) into the expressions for the Solution (90), and the Potential (91), taking into account that $n = 2$. The results are long and involved, so we refrain from presenting them in their general form here. Instead, we show specific cases of the solution $\hat{\Phi}$ to our transformed azimuthal Equation (92), that we compute by means of (93) and (94). Let us make the overall settings:

$$\nu_1 = 1 \quad \nu_2 = 2 \quad N = N_1 = 0\,. \tag{101}$$

We can now demonstrate specific cases of our Solution (94). The two solutions exhibiting the simplest form are given by:

$$\hat{\Phi}(\phi)_{r_1=r_2=1} = \frac{\cos(\phi)^5 \sin(\phi)^2 \, {}_2F_1\left[2,5,\frac{9}{2},\sin(\phi)^2\right]}{-3\frac{\arcsin[\sin(\phi)]}{\sin(\phi)^5} + \cos(\phi)\left[-8 + \frac{2}{\sin(\phi)^2} + \frac{3}{\sin(\phi)^4}\right]}$$

$$\hat{\Phi}(\phi)_{r_1=r_2=-1} = \frac{\cos(\phi)^8 \sin(\phi)^3 \, {}_2F_1\left[2,7,\frac{11}{2},\sin(\phi)^2\right]}{-15\frac{\arcsin[\sin(\phi)]}{\sin(\phi)^7} + \cos(\phi)\left[-112 - 64\cos(2\phi) + \frac{8}{\sin(\phi)^2} + \frac{10}{\sin(\phi)^4} + \frac{15}{\sin(\phi)^6}\right]}.$$

Graphs of these solutions and of more general cases are shown in the Figures 8 and 9. Note that for the sake of brevity, we will not present the explicit conditions for the parity constants r_1 and r_2.

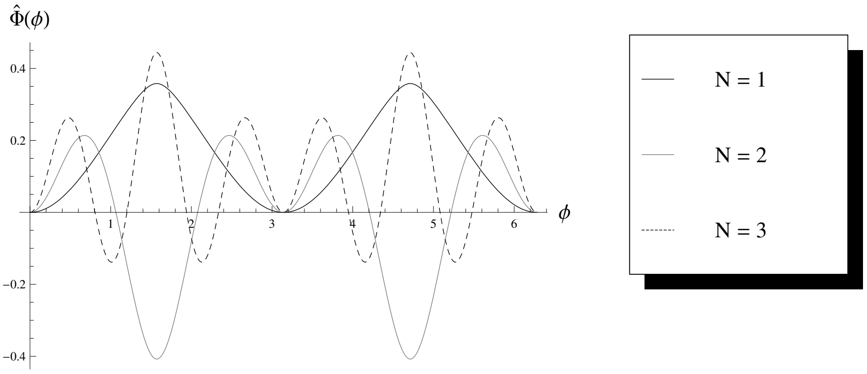

Figure 8. Graphs of the Solution (80) for the parameter Settings (101) and $r_1 = r_2 = 1$ for different values of N.

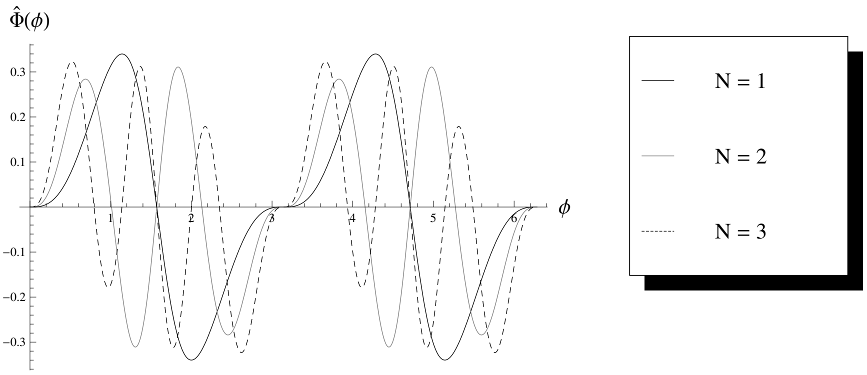

Figure 9. Graphs of the Solution (80) for the parameter Settings (101) and $r_1 = r_2 = -1$ for different values of N.

Before we conclude this application, let us present examples of the Potential (95) that is inserted in the transformed azimuthal Equation (92). Since the transformation Function (100) is not elementary, the potential cannot be expressed through elementary functions either. Since its general form is very long, we will not show it here. Instead, we show examples in Figure 10, where we observe that these potentials can be interpreted as deformed trigonometric Pöschl-Teller interactions. Note that this behaviour is expected due to the form of (95).

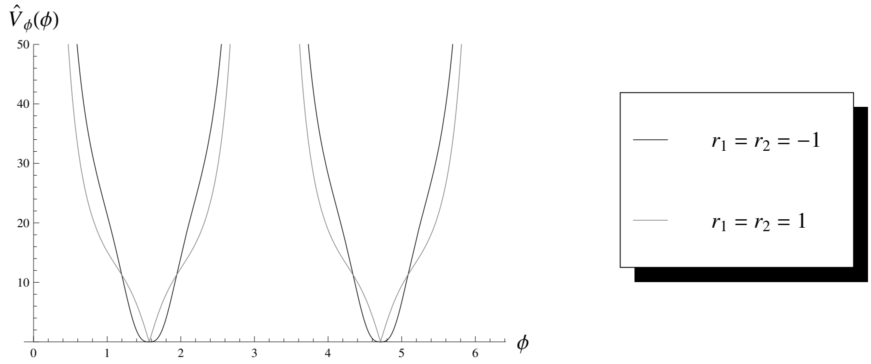

FIGURE 10. Graphs of the Potential (80) for the parameter Settings (101) and different values of r_1 and r_2.

6. Concluding remarks

The main observation in this work concerns the close interrelationship between the angular equations associated with three-dimensional, non-relativistic Dunkl-Schrödinger equations in spherical coordinates that is derived in Section 3, and one-dimensional trigonometric Pöschl-Teller systems. This interrelationship can be seen after separating the variables in the three-dimensional equation. As shown in Sections 4.1 and 5.1, a simple point transformation takes each of the two angular equations resulting from the separation, into the form of a Schrödinger equation for a trigonometric Pöschl-Teller potential. It is interesting to note here that in the standard case without the presence of Dunkl operators $\nu_1 = \nu_2 = \nu_3 = 0$, the two angular Equations (27) and (28) do not contain a trigonometric Pöschl-Teller potential. More precisely, the polar Equation (27) maintains a term proportional to an inverse sine function, while there is no trigonometric term in the azimuthal Equation (28). Returning to the Dunkl scenario, we can find solutions of our angular equations by using known results on trigonometric Pöschl-Teller systems, as demonstrated in Sections 4.1 and 5.1, respectively. Since the latter angular equations in their versions (30) and (73) match the Schrödinger form, the Darboux-Crum transformation becomes applicable to them. As the examples in Sections 4.2 and 5.2 illustrate, the application of the Darboux-Crum transformation allows to construct solvable angular equations within the Dunkl formalism by using results on trigonometric Pöschl-Teller systems and their generalisations. As mentioned above, the Darboux-Crum transformation generates potentials that will generally depend on all parameters in the respective angular equation except for the transformation parameter. Since this behaviour is undesirable, it is necessary to assign fixed values to all these parameters before performing the Darboux-Crum transformation, which imposes a strong constraint on the transformed system.

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