Return Words and Recurrence Function of a Class of Infinite Words

L. Balková

Many combinatorial and arithmetical properties have been studied for infinite words $u_\beta$ associated with $\beta$-integers. Here, new results describing return words and recurrence function for a special case of $u_\beta$ will be presented. The methods used here can be applied to more general infinite words, but the description then becomes rather technical.

Keywords: return words, recurrence function, Parry numbers.

1 Introduction

The study of factor complexity, palindromic complexity, balance property, return words, and the recurrence function of infinite aperiodic words is an interesting combinatorial problem. Moreover, the investigation of infinite words coding $\beta$-integers $\mathbb{Z}_\beta$, for $\beta$ being a Pisot number, can be interpreted as an investigation of one-dimensional quasicrystals. In this paper, new results concerning return words and recurrence function will be presented. In general, little is known about return words. A return word of factor $w$ of an infinite word $u$ is any word which starts at some occurrence of $w$ in $u$ and ends just before the next occurrence of $w$. Vuillon [1] has shown that an infinite word over a bilateral alphabet is sturmian if and only if each of its factors has exactly two return words. Justin and Vuillon [2] proved that each factor of an Arnoux-Rauzy word of order $m$ has $m$ return words. Moreover, Vuillon proved that each factor of an -interval exchange coding word also has return words. Return words in the Thue-Morse sequence and return words in the infinite words associated with simple Parry numbers have been studied in [4]. Even less is known about the recurrence function which, for an infinite uniformly recurrent word $u$, determines the minimal length $R_f(n)$ such that if we take any subword of length $R_f(n)$ of $u$, we will find there all the factors of length $n$. Cassaigne [5] determined this function for sturmian words taking into account the continued fractions of their slope and, in [6], he given a general algorithm describing how to determine the recurrence function if we know the return words. This algorithm will constitute the headstone of our further results.

We will describe the cardinality of the set of return words for any factor $w$ of the infinite word $u_\beta$ associated with quadratic non-simple Parry (and, thus Pisot) number $\beta$ and, as a consequence, we will be able to deduce the exact formula for the recurrence function. These results will complete the list of properties already studied: factor complexity has been studied in [7], palindromic complexity is described in [8], and results on balances can be found in [9].

2 Preliminaries

First, let us introduce our “language”, which will be used throughout this paper. An alphabet $A$ is a finite set of symbols called letters. A concatenation of letters is a word. The length of word $w$ is denoted by $|w|$. The set $A^*$ of all finite words (including the empty word $\varepsilon$) provided with the operation of concatenation is a free monoid. We will also deal with right-sided infinite words $u = u_0u_1u_2 \ldots$. A finite word $w$ is called a factor of the word $u$ (finite or infinite) if there exist a finite word $u^{(1)}$ and a word $u^{(2)}$ (finite or infinite) such that $u = w^{(1)}u^{(2)}$. The word $w$ is a prefix of $u$ if $u^{(1)} = \varepsilon$ and it is a suffix of $u$ if $u^{(2)} = \varepsilon$. A concatenation of $k$ letters $a$ (or words $a$) will be denoted by $a^k$. The language of $u$, $L(u)$, is the set of all factors of a word $u$, and $L_n(u)$ is the set of all factors of $u$ of length $n$. We say that a letter $a \in A$ is a left extension of a factor $w \in L(u)$ if the factor $aw$ belongs to the language $L(u)$ and $w$ is left special if $w$ has at least two left extensions. Right extensions and right special factors are defined by analogy. We call a factor $w \in L(u)$ bispecial if it is left special and right special. A mapping $\phi$ on the free monoid $A^*$ is called a morphism if $\phi(wv) = \phi(w)\phi(v)$ for all $w, v \in A^*$. Obviously, for determining any morphism it suffices to give $\phi(a)$ for all $a \in A$. The action of a morphism can be naturally extended on right-sided infinite words by the prescription

$$\phi(u_0u_1u_2 \ldots) := \phi(u_0) \phi(u_1) \phi(u_2) \ldots$$

A non-erasing morphism $\phi$, for which there exists a letter $a \in A$ such that $\phi(a) = aw$ for some non-empty word $w \in A^*$, is called a substitution. An infinite word $w$ such that $\phi(u) = u$ is called a fixed point of the substitution $\phi$. Obviously, every substitution has at least one fixed point, namely

$$\lim_{n \to \infty} \phi^n(a).$$

In everything that follows, we will focus on the infinite word $u_\beta$ being the only fixed point of the substitution $\phi$ given by

$$\phi(0) = 0^b1, \quad \phi(1) = 0^{a-1}b, \quad a \geq b, \quad a \in \mathbb{N}. \quad (1)$$

Let us only briefly mention that this infinite word codes the set of $\beta$-integers $\mathbb{Z}_\beta$, where $\beta$ is a quadratic non-simple Parry number. All details can be found, for example, in [8]. Since our main interest is in studying return words and the recurrence function, let us give the corresponding definitions.

Definition 1

Let $w$ be a factor of an infinite word $u = u_0u_1 \ldots \ldots (\text{with } u_j \in A)$, $|w| = \ell$. An integer $\phi$ is a factor of $w$ in $u$ if $u_j, u_{j+1}, \ldots, u_{j+\ell-1} = w$.

Let $j, k, j \leq k$, be successive occurrences of $w$. Then $u_{j+k+1} \ldots u_{k+1}$ is a return word of $w$. The set of all return words of $w$ is denoted by $\text{Ret}(w)$, i.e.,

$$\text{Ret}(w) = \{u_ju_{j+1} \ldots u_{k-1} | j, k \text{ being successive occurrences of } w\}.$$
An infinite word $u$ is said to be recurrent if each of its factors appears infinitely many times in $u$. It is said to be uniformly recurrent if for every $n \in \mathbb{N}$ there exists $R(n) > 0$ such that any segment of $u$ of length $\geq R(n)$ contains all words in $L_n(u)$.

As we deal with the infinite word $u_g = \lim_{n \to \infty} \varphi(u)$, where $\varphi(0) = 0^a1, \varphi(l) = 0^{a_1}1, \ldots, a_i \geq b$, let us also recall what is known about the uniform recurrence of words being fixed points of a substitution.

**Definition 2**

A substitution $\varphi$ over an alphabet $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ is called primitive if there exists $K \in \mathbb{N}$ such that for any letter $a_i \in \mathcal{A}$, the word $\varphi^K(a_i)$ contains all letters of $\mathcal{A}$.

Queffélec in [10] has shown that any fixed point of a primitive substitution, $u_g$ is a uniformly recurrent word. Thus, since the infinite word $ub$ is a fixed point of a primitive substitution, $u_b$ is a uniformly recurrent word. It is not difficult to see that the set of return words of $w$ is finite for any factor $w \in \mathcal{L}(u)$ if $u$ is a uniformly recurrent word.

**Definition 3**

The recurrence function of an infinite word $u$ is the function $R_n: \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by

$$R_n(u) = \inf \left\{ N \in \mathbb{N} | \forall v \in \mathcal{L}_N(u), \mathcal{L}_n(v) = \mathcal{L}_n(u) \cup \{+\infty\} \right\}$$

In other words, $R_n(u)$ is the smallest length such that if we take any segment of $u$ of length $R_n(u)$, we will find there all the factors of $u$ of length $n$. Clearly, $u$ is uniformly recurrent if and only if $R_n(u)$ is finite for every $n \in \mathbb{N}$. Let us also recall what is known about the uniform recurrence of words being fixed points of a substitution.

**Proposition 5**

For any infinite uniformly recurrent word $u$ and for any $n \in \mathbb{N}$, one has

$$R_n(u) = l_w(n) + n - 1$$

3 Return words

The aim of this section is to determine return words of the infinite word $u_g$ being the fixed point of the substitution $\varphi$ introduced in (1). Vuillon in [1] has shown the following result for sturmian words.

**Proposition 6**

Let $u$ be an infinite word over a two letter alphabet. Then $u$ is sturmian if and only if $\# \text{Ret}(u) = 2$ for every factor $w$ of $u$.

Let us mention that $u_g$ is sturmian for $a - 1 = b$, as follows from [7].

**Example 7**

Let $u_g = 0010010010010010101 \ldots$ be the fixed point of the substitution $\varphi(0) = 001, \varphi(l) = 01$ (sturmian case). Let us show examples of return words:

$$\text{Ret}(0) = \{0, 01\}$$
$$\text{Ret}(00) = \{001, 00101\}$$
$$\text{Ret}(001) = \{001, 00101\}$$
$$\text{Ret}(0010) = \{011, 00101\}$$

Throughout this section, we will use methods analogous to those ones introduced in [4]. In order to study return words $\text{Ret}(w)$ of factors $w$ of an infinite uniformly recurrent word $u$, it is possible to limit our considerations to bispecial factors. Namely, if a factor $w$ is not right special, i.e., if it has a unique right extension $a \in \mathcal{A}$, then the sets of occurrences of $w$ and $bw$ coincide, and

$$\text{Ret}(w) = \text{Ret}(bw)$$

If a factor $w$ has a unique left extension $b \in \mathcal{A}$, then $j \geq 1$ is an occurrence of $w$ in the infinite word $u$ if and only if $j - 1$ is an occurrence of $bw$. This statement does not hold for $j = 0$. Nevertheless, if $u$ is a uniformly recurrent infinite word, then the set $\text{Ret}(w)$ of return words of $w$ stays the same no matter whether we include the return word corresponding to the prefix $w$ of $w$ or not. Consequently, we have $\text{Ret}(bw) = b\text{Ret}(w)$. This means that the word $v$ is extended to the left by the letter $b$ and the letter $b$ on the right end is erased (Note that $b$ is always the suffix of $v$ for $v \in \text{Ret}(w)$).

**Observation 8**

The only bispecial factors of $u_g$ that do not contain the letter 1 are $0^r$, $r \leq a - 1$.

**Observation 10**

Every bispecial factor of $w$ of $u_g$ containing at least one letter 1 has the prefix $0^{a-1}$ and the suffix $0^b$. Consequently, there exists a bispecial factor $v$ such that $w = 0^a\varphi(v)0^b$. (The empty word $\epsilon$ is bispecial, too.)

Let us summarize the previous observations to get the description of all bispecial factors.

**Corollary 11**

The set of all the bispecial factors of $u_g$ is given by

$$B_r^{(n)} = \{ b^{n+1} \varphi(b_r^{(n)})^0 | n \in \mathbb{N} \}$$

where $B_r^{(n+1)} = 0^b \varphi(B_r^{(n)})^0$ for all $n \in \mathbb{N}$, $B_0^{(1)} = 0^a$, and $B_r^{(1)} = 0^b$ for $r \in \{0, \ldots, a - 1\}$.

In order to find return words of bispecial factors, let us recall a lemma from [4]. To formulate it, we also need to recall a definition.

**Definition 12**

Let $w$ be a factor of a fixed point $u$ of a substitution $\varphi$. We say that a word $v_0^a v_1^b \ldots v_{n-1}^a v_n^b$ in $\mathcal{L}_n(u)$ is an ancestor of $w$ if

- $w$ is a factor of $\varphi(v_0^a v_1^b \ldots v_{n-1}^a v_n^b)$.
Observation 17
is a sturmian word
bispecial factors are the following:
Let w be a factor of u
Then, Relations (2) and (3) say that
Let w be a non-empty factor of u
Proposition 14
Let an infinite word u be a fixed point of a substitution
Lemma 13
Let an infinite word u be a fixed point of a substitution
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
Proof
4 How to compute the recurrence function
Let us show how to apply the knowledge of return words to
describe the recurrence function (Definition 3) of the infinite
word uβ. The methods used here follow the ideas from [6].
As the reader will have anticipated, we want to compute
lβ(n) in order to get the recurrence function Rn(u) for every
n ∈ N. This task can be simplified using the notion of singular
factors. A factor w ∈ L(u) is called singular if |w| = 1 or if
there exist a word v ∈ L(u) and letters x, y, y′ ∈ A such that
w = xy, x ≠ x′, y ≠ y′, and {x′, yy′} ⊂ L(u). Obviously, v is a
bispecial factor.
Proposition 18
Let u be a uniformly recurrent word and n ≥ 1. If lβ(n − 1) < lβ(n),
then there exists a singular factor w of length n such that
lβ(occur) = lβ(n).
A singular factor w is said to be essential if
lβ(occur) = lβ(|w|) > lβ(|w|) − 1.
It follows that to calculate lβ(n), it is sufficient to consider
singular, or, even, only essential singular factors.
Theorem 19
Let u be a uniformly recurrent word and n ≥ 1.
lβ(n) = max{lβ(w)|w| ≤ n and w singular} = max{lβ(w)|w| ≤ n and w essential singular}.
Now we are able to give an algorithm for computing the recurrence function of an infinite uniformly recurrent
word u:
1. Determine bispecial factors.
2. Deduce the form of singular factors and compute their
lengths.
3. For every singular factor, determine the associated return
words and compute their lengths.
4. Compute the function lβ(n) to get the recurrence function
Rn(u) for every n ∈ N using Proposition 5.
5 Computation of the recurrence function of uβ
Let us apply the above described algorithm for computing the recurrence function of the infinite word uβ being the only
fixed point of the substitution ϕ(0) = 0, ϕ(1) = 0, a − 1 ≥ b.
1. The bispecial factors of uβ are described in Corollary 11.
2. Let us show how the task of describing the singular factors
can be simplified using the relation between a factor and
its image under the substitution ϕ.
Observation 20
Let n ≥ 2 and r ∈ {0, ..., a − 1}. Then, S(n)(x, y) = x B(n−1) y is a
singular factor if and only if S(n−1)(x, y) = x B(n−1) y is a singular
factor.
If we describe all the simplest non-trivial singular factors, i.e., those factors of the form S(n−1)(x, y) = x B(n−1) y,
x, y ∈ {0, 1}, Observation 20 will allow us to get the set of all
singular factors of uβ.
Proposition 21
The set of all singular factors of $u_{xy}$ is given by
$$\{0,1\} \cup \{S_{y}^{(n)}(x,y)\; |\; r \in \{0,\ldots, a-1\}, \; n \in \mathbb{N}\}$$
where $S_{y}^{(n)}(x,y) = x S_{y}^{(n-1)}(y,1)$ for all $n \in \mathbb{N}$ and $x, y$ come from
the form of the simplest non-trivial singular factors:
$$S_{r}^{(0)}(0,0) = 00^a, \; r \geq a - 2$$
$$S_{r}^{(0)}(1,0) = 0^b 0, \; S_{r}^{(0)}(0,1) = 0^b 1, \; S_{r}^{(0)}(1,1) = 10^b,$$
$$S_{0}^{(0)}(0,0) = 00.$$  

To compute the length of singular factors, it is, of course, enough to compute the lengths of bispecial factors since $S_{y}^{(n)}(x,y) = |S_{y}^{(n)}| + 2$. For the lengths of bispecial factors, we have $|S_{y}^{(n)}| = |S_{y}^{(n-1)}| + |S_{y}^{(n)}|$, where $n$ denotes the number of 0’s in $S_{y}^{(n)}$ and $|S_{y}^{(n)}|$ denotes the number of 1’s in $S_{y}^{(n)}$. Then
$$|S_{y}^{(n)}| = \left(\begin{array}{c} |S_{y}^{(n-1)}| \\ 0 \\ \frac{a}{b} \end{array}\right), \quad \text{for all } n \geq 2.$$  

3. To describe the return words of singular factors, we can again make use of a relation between return words of a singular factor and return words of its image under $f$.  

Proposition 22
Let $n \geq 2, r \in \{0,\ldots, a-1\}$. The following sets are equal
$$\{w \in \text{Ret}(S_{y}^{(n)}(x,y))\} = \{f(w) \in \text{Ret}(S_{r}^{(n-1)}(x,y))\}.$$  

Proof. We will distinguish two situations.

a) For $x = 0$ and $y \in \{0,1\}$, applying Observation 8 on return words of factors which are not right or left special, it follows that the set of lengths of return words of $S_{y}^{(n)}(x,y) = x_0^{|S_{y}^{(n-1)}|} f(S_{y}^{(n-1)}(y,1))$ is the same as that of $\varphi(x) \varphi(S_{y}^{(n-1)}(y,1))$. Thus, we get
$$\{w \in \text{Ret}(S_{y}^{(n)}(x,y))\} = \{f(w) \in \text{Ret}(\varphi(S_{y}^{(n-1)}(y,1)))\}.$$  

b) The singular factor $S_{y}^{(n)}(1,1) = 10^b 1 f(S_{y}^{(n-1)}(1,1))$ can be extended to the left without changing the set of lengths of return words to
$$0^{|S_{y}^{(n-1)}|} 1 f(S_{y}^{(n-1)}(1,1)) = \varphi(0) 1 \varphi(S_{y}^{(n-1)}(1,1)).$$  

Thus, we get
$$\{w \in \text{Ret}(S_{y}^{(n)}(x,y))\} = \{f(w) \in \text{Ret}(\varphi(S_{y}^{(n-1)}(1,1)))\}.$$  

The statement then follows using Lemma 12 and Proposition 14, and, in the case of (b), also using Relation (3) for return words of $0S_{y}^{(n-1)}(1,1)$ and $S_{y}^{(n-1)}(1,1)$.  

Now it suffices to determine return words and their lengths for the simplest singular factors $S_{r}^{(1)}(x,y)$. Proposition 22 implies that the lengths of return words of all the other singular factors will be obtained by calculating the lengths of images of the simplest return words. Here are the return words of the simplest singular factors.

a) For the trivial singular factors 0, 1, one gets
$$\text{Ret}(0) = \{0,0\} \quad \text{and} \quad \text{Ret}(1) = \{10^b, 1\}.$$  

b) For $S_{r}^{(1)}(0,0) = 00^a$, using the proof of Theorem 16, we have
$$\text{Ret}(00^a) = \begin{cases} \{0,0\} & \text{if } r = a - 2 \\ \{0,0^2, 0^21\} & \text{if } b - 2 < r < a - 2 \\ \{0,0^21\} & \text{if } 1 \leq r \leq b - 2. \end{cases}$$  

c) For $S_{r}^{(1)}(1,0) = 10^b 0$, one can easily find
$$\text{Ret}(10^b 0) = \{10^b, 10^21\}.$$  

d) For $S_{r}^{(1)}(0,1) = 00^b 1$, one has
$$\text{Ret}(00^b 1) = \{0,0^b, 0^b1, 0^b10^b, 0^b10^b1\}.$$  

e) Since $S_{r}^{(1)}(1,1) = 10^b 1 1 = 1 \times \varphi(1)$, using Relation (3), we have
$$\text{Ret}(10^b 1) = \{10^b, 10^b1\}.$$  

f) For $S_{r}^{(1)}(0,0) = 00$, one has to consider more cases:
- Let $b = 1$ and $a = 2$, then $\text{Ret}(00) = \{00, 010\}.$
- Let $b = 1$ and $a > 2$, then $\text{Ret}(00) = \{00, 010, 10\}.$
- Let $b \geq 2$, then $\text{Ret}(00) = \{00, 10\}.$  

4. The last step is to compute $l_{a_{xy}}(k), k \in \mathbb{N}$. For simplicity, let us write $l(k)$ instead of $l_{a_{xy}}(k)$ in the sequel. Before starting, let us exclude singular factors which are not essential. (Let us recall that a singular factor $w$ is essential if $|f(w)| = k |w| > |w| - 1$.) Naturally, we will apply Proposition 22 to determine the maximal return times $l(w)$ of singular factor $w$.

a) Note that $l_{b_{y}}(1,0,0) = l_{b_{y}}(0,1,1) = l_{b_{y}}(1,1)$, but from Relations (6), (7), and (8), it is clear that for all $n \in \mathbb{N}$, one gets $l_{b_{y}}(1,0) < l_{b_{y}}(0,1) < l_{b_{y}}(1,1)$. Thus, $S_{b_{y}}(1,0)$ and $S_{b_{y}}(0,1)$ are not essential singular factors and one does not have to consider them in calculation of $l(k), k \in \mathbb{N}$.

b) By analogy, if $b \geq 2$, we have $l_{b_{y}}(0) > l_{b_{y}}(1,1)$ while $l_{b_{y}}(0) < l_{b_{y}}(1,1) < l_{b_{y}}(1,1)$ for all $n \in \mathbb{N}$. Hence, the singular factors $S_{b_{y}}(0,0) = 00$ are not essential.

c) Next, if $a \geq 2$, we have $S_{r}^{(n)}(0,0) = S_{r}^{(n)}(1,1)$ nevertheless, $l_{b_{y}}(0) < l_{b_{y}}(1,1)$ for all $n \in \mathbb{N}$. Therefore, $S_{r}^{(n)}(0,0), r \geq b, a$ are not essential.
d) If \( 1 \leq r \leq b - 2 \), then we obtain \( \left| S_r^{(n+1)}(0,0) \right| > \left| S_r^{(n)}(1,1) \right| \), but \( l(S_r^{(n+1)}(0,0)) < l(S_b^{(n)}(1,1)) \) for all \( n \in \mathbb{N} \), thus, the singular factors \( S_r^{(n+1)}(0,0) \), \( 1 \leq r \leq b - 2 \), are not essential.

e) The last remark is that for the trivial singular factor \( w = 1 \), we have
\[
\left| S_r^{(1)}(0,0) \right| \leq 1 \leq b - 2,
\]
but \( l(S_r^{(1)}(0,0)) < l(w) \), hence, \( S_r^{(1)}(0,0) \), \( 1 \leq r \leq b - 2 \), are not essential.

The previous facts imply that in order to calculate \( l(k) \) we have to take into account only the trivial singular factors, the non-trivial singular factors of the form \( S_r^{(n)}(1,1) \), and, eventually, \( S_r^{(n)}(0,0) \) and \( S_r^{(n)}(0,0) \). The formulae for \( l(k) \), \( k \in \mathbb{N} \), will split into more cases, according to the values of \( a \) and \( b \) in the substitution (1).

a) For \( b \geq 2 \), combining the previous facts and the description of the simplest singular factors, we obtain the following formula for \( l(k) \), \( k \in \mathbb{N} \),
\[
\begin{align*}
l(1) = & \ldots = l(b) = a + 1, \\
l(b + 1) = & 2b + 3, \\
l(k) = & p^n(10^b) \\
& \text{for } a + 1 < k \leq 60, \\
l(k) = & p^n(10^{b+1}) \\
& \text{for } 60 < k \leq 10^n, \\
l(k) = & p^n(00101) \\
& \text{for } k \leq 3, \\
l(k) = & p^n(0011) \\
& \text{for } 3 < k \leq 3, \\
l(k) = & p^n(110) \\
& \text{for } 110 < k \leq 111, \\
l(k) = & p^n(111) \\
& \text{for } 111 < k \leq 111.
\end{align*}
\]

b) For \( b = 1 \) and \( 2 \leq a \leq 3 \), then it holds for \( l(k) \), \( k \in \mathbb{N} \),
\[
\begin{align*}
l(1) = & a + 1, \\
l(2) = & 5, \\
l(k) = & p^n(10^b) \\
& \text{for } a + 1 < k \leq 60, \\
l(k) = & p^n(00101) \\
& \text{for } 60 < k \leq 3, \\
l(k) = & p^n(0011) \\
& \text{for } 3 < k \leq 3, \\
l(k) = & p^n(110) \\
& \text{for } 110 < k \leq 111, \\
l(k) = & p^n(111) \\
& \text{for } 111 < k \leq 111.
\end{align*}
\]

The table of the first 20 values of \( l(k) \) and \( R(k) \) for the simplest case \( a = 2 \) and \( b = 1 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l(k) )</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>8</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>34</td>
<td>34</td>
<td>34</td>
<td>34</td>
<td>34</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>( R(k) )</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>11</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
<td>32</td>
<td>46</td>
<td>47</td>
<td>48</td>
<td>49</td>
<td>50</td>
<td>51</td>
<td>52</td>
<td></td>
</tr>
</tbody>
</table>

\(\) References


Ing Lúbomíra Balková
phone: +420 224 358 560
e-mail: l.balkova@centrum.cz

Doppler Institute for Mathematical Physics and Applied Mathematics & Department of Mathematics
Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Trojanova 13
120 00 Praha 2, Czech Republic