Self-Matching Properties of Beatty Sequences

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We study the selfmatching properties of Beatty sequences, in particular of the graph of the function $\lfloor j\beta \rfloor$ against j for every quadratic unit $\beta \in (0,1)$. We show that translation in the argument by an element G_i of a generalized Fibonacci sequence almost always causes the translation of the value of the function by G_{i-1}. More precisely, for fixed $i \in \mathbb{N}$, we have $\lfloor \beta(j+G_i) \rfloor = \lfloor \beta j \rfloor + G_{i-1}$, where $j \in U_i$. We determine the set U_i of mismatches and show that it has a low frequency, namely β^i.

Keywords: Beatty sequences, Fibonacci numbers, cut-and-project scheme.

1 Introduction

Sequences of the form $\left(\lfloor j\alpha \rfloor\right)_{j\in\mathbb{N}}$ for $\alpha > 1$, now known as Beatty sequences, were first studied in the context of the famous problem of covering the set of positive integers by disjoint sequences [1]. Further results in the direction of so-called disjoint covering systems are due to [2], [3], [4] and others. Other aspects of Beatty sequences were then studied, such as their generation using graphs [5], their relation to generating functions [6], [7], their substitution invariance [8], [9], etc. A good source of references on Beatty sequences and other related problems can be found in [10], [11].

In [12] the authors study the self-matching properties of the Beatty sequence $\left(\lfloor j\tau \rfloor\right)_{j\in\mathbb{N}}$ for $\tau = \frac{1}{2}(\sqrt{5}-1)$, the golden ratio. Their study is rather technical; they have used for their proof the Zeckendorf representation of integers as a sum of distinct Fibonacci numbers. The authors also state an open question whether the results obtained can be generalized to other irrationals than τ. In our paper we answer this question in the affirmative. We show that Beatty sequences $\left(\lfloor j\alpha \rfloor\right)_{j\in\mathbb{N}}$ for quadratic Pisot units α have a similar self-matching property, and for our proof we use a simpler method, based on the cut-and-project scheme.

It is interesting to note that Beatty sequences, Fibonacci numbers and the cut-and-project scheme have attracted the attention of physicists in recent years because of their applications for mathematical description of non-crystallographic solids with long-range order, so-called quasicrystals, discovered in 1982 [13]. The first observed quasicrystals revealed crystallographically forbidden rotational symmetry of order 5. This necessitates, for an algebraic description of the mathematical model of such a structure, the use of the quadratic field $\mathbb{Q}(\tau)$. Such a model is self-similar with the scaling factor τ^{-1}. Later, the existence was observed of quasicrystals with 8 and 12-fold rotational symmetries, corresponding to mathematical models with selfsimilar factors $\mu^{-1} = 1+\sqrt{2}$ and $\nu^{-1} = 2+\sqrt{3}$. Note that all τ, μ, and ν are quadratic Pisot units, i.e. they belong to the class of numbers for which the result of Bunder and Tognetti is generalized here.

2 Quadratic Pisot units and the cut-and-project scheme

The self-matching properties of the Beatty sequence $\left(\lfloor j\tau \rfloor\right)_{j\in\mathbb{N}}$ are best displayed on the graph of $\lfloor j\tau \rfloor$ against $j \in \mathbb{N} = \{1,2,3,\ldots\}$. An important role is played by the Fibonacci numbers,

$$F_0 = 0, F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}, \text{ for } k \geq 1\,.$$

The result of [12] states that

$$\lfloor (j+F_i)\tau \rfloor = \lfloor j\tau \rfloor + F_{i-1}, \tag{1}$$

with the exception of isolated mismatches of frequency τ^i, namely at points of the form $j = kF_{i+1} + \lfloor k\tau \rfloor F_i$, $k \in \mathbb{N}$.

Our aim is to show a very simple proof of these results that is valid for all quadratic units $\beta \in (0,1)$. Every such unit is a solution of the quadratic equation

$$x^2 + mx = 1, \quad m \in \mathbb{N},$$

or

$$x^2 - mx = -1, \quad m \in \mathbb{N}, m \geq 3.$$

The considerations will differ slightly in the two cases.

a) Let $\beta \in (0,1)$ satisfy $\beta^2 + m\beta = 1$ for $m \in \mathbb{N}$. The algebraic conjugate β' of β, i.e. the other root β' of the equation, satisfies $\beta' > -1$. We define the generalized Fibonacci sequence

$$G_0 = 0, \; G_1 = 1, \quad G_{n+2} = mG_{n+1} + G_n\,, \; n \geq 0 \tag{2}$$

It is easy to show by induction that for $i \in \mathbb{N}$, we have

$$(-1)^{i+1}\beta^i = G_i\beta - G_{i-1} \text{ and } (-1)^{i+1}\beta'^i = G_i\beta' - G_{i-1}. \tag{3}$$

b) Let $\beta \in (0,1)$ satisfy $\beta^2 - m\beta = -1$ for $m \in \mathbb{N}$, $m \geq 3$. The algebraic conjugate β' of β satisfies $\beta' > 1$. We define

$$G_0 = 0, \; G_1 = 1, \quad G_{n+2} = mG_{n+1} - G_n\,, \; n \geq 0 \tag{4}$$

In this case, we have for $i \in \mathbb{N}$

$$\beta^i = G_i\beta - G_{i-1} \quad\text{and}\quad \beta'^i = G_i\beta' - G_{i-1} \qquad (5)$$

The proof we give here is based on the algebraic expression for one-dimensional cut-and-project sets [14]. Let V_1, V_2 be straight lines in $\mathbb{R}^2$ determined by vectors $(\beta, -1)$ and $(\beta', -1)$, respectively. The projection of the square lattice $\mathbb{Z}^2$ on the line V_1 along the direction of V_2 is given by

$$(a,b) = (a + b\beta')\vec{x}_1 + (a + b\beta)\vec{x}_2, \text{ for } (a,b) \in \mathbb{Z}^2,$$

where $\vec{x}_1 = \dfrac{1}{\beta - \beta'}(\beta, -1)$ and $\vec{x}_2 = \dfrac{1}{\beta' - \beta}(\beta', -1)$. For the description of the projection of $\mathbb{Z}^2$ on V_1 it suffices to consider the set

$$\mathbb{Z}[\beta'] := \{a + b\beta' \mid a, b \in \mathbb{Z}\}$$

The integral basis of this free abelian group is $(1, \beta')$, and thus every element x of $\mathbb{Z}[\beta']$ has a unique expression in this base. We will say that a is the rational part of $x = a + b\beta'$ and b is its irrational part. Since β' is a quadratic unit, $\mathbb{Z}[\beta']$ is a ring and, moreover, it satisfies

$$\beta'\mathbb{Z}[\beta'] = \mathbb{Z}[\beta'] \qquad (6)$$

A cut-and-project set is the set of projections of points of $\mathbb{Z}^2$ to V_1, that are found in a strip of given bounded width, parallel to the straight line V_1. Formally, for a bounded interval Ω we define

$$\Sigma(\Omega) = \{a + b\beta' \mid a, b \in \mathbb{Z}, a + b\beta \in \Omega\}$$

Note that $a + b\beta'$ corresponds to the projection of the point (a, b) to the straight line V_1 along V_2, whereas $a + b\beta$ corresponds to the projection of the same lattice point to V_2 along V_1.

Among the simple properties of the cut-and-project sets that we use here are

$$\Sigma(\Omega - 1) = -1 + \Sigma(\Omega), \qquad \beta'\Sigma(\Omega) = \Sigma(\beta\Omega),$$

where the latter is a consequence of (6). If the interval Ω is of unit length, one can derive directly from the definition a simpler expression for $\Sigma(\Omega)$. In particular, we have

$$\Sigma[0,1) = \{a + b\beta' \mid a + b\beta \in [0,1)\} = \{b\beta' - \lfloor b\beta \rfloor \mid b \in \mathbb{Z}\}, \qquad (7)$$

where we use that the condition $0 \le a + b\beta < 1$ is satisfied if and only if $a = \lceil -b\beta \rceil = -\lfloor b\beta \rfloor$.

Let us mention that the above properties of one-dimensional cut-and-project sets, and many others, are explained in the review article [14].

3 Self-matching property of the graph $\lfloor j\beta \rfloor$ against j

An important role in the study of the self-matching properties of the graph $\lfloor j\beta \rfloor$ against j is played by the generalized Fibonacci sequence $(G_i)_{i\in\mathbb{N}}$, defined by (2) and (4), respectively. It turns out that shifting the argument j of the function $\lfloor j\beta \rfloor$ by the integer G_i results in shifting the value by G_{i-1}, with the exception of isolated mismatches with low frequency. The first proposition is an easy consequence of the expressions of β' as an element of the ring $\mathbb{Z}[\beta]$ in the integral basis $1, \beta$, given by (3) and (5).

Theorem 1

Let $\beta \in (0,1)$ satisfy $\beta^2 + m\beta = 1$ and let $(G_i)_{i=0}^{\infty}$ be defined by (2). Let $i \in \mathbb{N}$. Then for $j \in \mathbb{Z}$ we have

$$\lfloor \beta(j + G_i) \rfloor = \lfloor j\beta \rfloor + G_{i-1} + \varepsilon_i(j)$$

where $\varepsilon_i(j) \in \{0, (-1)^{i+1}\}$. The frequency of integers j for which the value $\varepsilon_i(j)$ is non-zero is equal to

$$\rho_i := \lim_{n\to\infty} \frac{\#\{j \in \mathbb{Z} \mid -n \le j \le n, \varepsilon_i(j) \ne 0\}}{2n+1} = \beta^i.$$

Proof. The first statement is trivial. For, we have

$$\begin{aligned}\varepsilon_i(j) &= \lfloor \beta(j + G_i) \rfloor - \lfloor j\beta \rfloor - G_{i-1} = \lfloor j\beta - \lfloor j\beta \rfloor + \beta G_i - G_{i-1} \rfloor \\ &= \lfloor j\beta - \lfloor j\beta \rfloor + (-1)^{i+1}\beta^i \rfloor \in \{0, (-1)^{i+1}\}.\end{aligned} \qquad (8)$$

The frequency ρ_i is easily determined in the proof of Theorem 1. ❑

In the following theorem we determine the integers j for which $\varepsilon_i(j)$ is non-zero. From this, we easily derive the frequency of such mismatches.

Theorem 2

With the notation of Theorem 1, we have

$$\varepsilon_i(j) = \begin{cases} 0 & \text{if } j \notin U_i, \\ (-1)^{i+1} & \text{otherwise,} \end{cases}$$

where

$$U_i = \{kG_{i+1} + \lfloor k\beta \rfloor G_i \mid k \in \mathbb{Z}, k \ne 0\} \cup \left\{\frac{(-1)^{i-1}}{2} G_i\right\}.$$

Before starting the proof, let us mention that for i even, the set U_i can be written simply as

$$U_i = \{kG_{i+1} + \lfloor k\beta \rfloor G_i \mid k \in \mathbb{Z}\}.$$

For i odd, the element corresponding to $k = 0$ is equal to $-G_i$ instead of 0. The distinction according to the parity of i is necessary here, since unlike the paper [12], we determine the values of $\varepsilon_i(j)$ for $j \in \mathbb{Z}$, not only for.

Proof. It is convenient to distinguish two cases according to the parity of i.

- First let i be even. It is obvious from (8), that $\varepsilon_i(j) \in \{0, -1\}$ and

$$\varepsilon_i(j) = -1 \quad \text{if and only if} \quad j\beta - \lfloor j\beta \rfloor \in [0, \beta^i). \qquad (9)$$

Let us denote by M the set of all such j,

$$\begin{aligned} M &= \{j \in \mathbb{Z} \mid j\beta - \lfloor j\beta \rfloor \in [0, \beta^i)\} \\ &= \{j \in \mathbb{Z} \mid k + j\beta \in [0, \beta^i), \text{ for some } k \in \mathbb{Z}\}\end{aligned}$$

Therefore M is formed by the irrational parts of the elements of the set

$$\left\{k + j\beta' \,\middle|\, k + j\beta \in [0, \beta^i)\right\} = \Sigma[0, \beta^i) = \beta'^i \Sigma[0,1)$$
$$= (-\beta' G_i + G_{i-1})\left\{k\beta' - \lfloor k\beta \rfloor \middle| k \in \mathbb{Z}\right\},$$

where the last equality follows from (3) and (7). Separating the irrational part we obtain

$$M = \left\{k G_i m + k G_{i-1} + \lfloor k\beta \rfloor G_i \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{G_i \lfloor k\beta \rfloor + k G_{i+1} \middle| k \in \mathbb{Z}\right\} = U_i,$$

where we have used the equations $\beta'^2 + m\beta' = 1$ and $mG_i + G_{i-1} = G_{i+1}$.

- Now let i be odd. Then from (8), $\varepsilon_i(j) \in \{0, -1\}$ and

$$\varepsilon_i(j) = 1 \quad \text{if and only if} \quad j\beta - \lfloor j\beta \rfloor \in [1 - \beta^i, 1). \qquad (10)$$

Let us denote by M the set of all such j,

$$M = \left\{ j \in \mathbb{Z} \middle| j\beta - \lfloor j\beta \rfloor - 1 \in [-\beta^i, 0)\right\}$$
$$= \left\{ j \in \mathbb{Z} \middle| k + j\beta \in [-\beta^i, 0), \ \text{for some } k \in \mathbb{Z}\right\}.$$

Therefore M is formed by the irrational parts of elements of the set

$$\left\{k + j\beta' \,\middle|\, k + j\beta \in [-\beta^i, 0)\right\} = \Sigma[-\beta^i, 0) = \beta'^i \Sigma[-1, 0)$$
$$= \beta'^i (1 - \Sigma[0,1)) = (\beta' G_i - G_{i-1})\left\{k\beta' - \lfloor k\beta \rfloor - 1 \middle| k \in \mathbb{Z}\right\}.$$

Separating the irrational part we obtain

$$M = \left\{-k G_i m - k G_{i-1} - \lfloor k\beta \rfloor G_i - G_i \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{-k G_{i+1} - G_i(\lfloor k\beta \rfloor + 1) \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{k G_{i+1} + G_i(\lceil k\beta \rceil - 1) \middle| k \in \mathbb{Z}\right\} = U_i,$$

where we have used the equation

$$\beta'^2 + m\beta' = 1, \ mG_i + G_{i-1} = G_{i+1} \ \text{and} \ -\lfloor k\beta \rfloor = \lceil k\beta \rceil.$$

Let us recall that the Weyl theorem [15] states that numbers of the form $j\alpha - \lfloor j\alpha \rfloor$, $j \in \mathbb{Z}$, are uniformly distributed in (0, 1) for every irrational α. Therefore the frequency of those $j \in \mathbb{Z}$ that satisfy $j\alpha - \lfloor j\alpha \rfloor \in I \subset (0,1)$ is equal to the length of the interval I. Therefore one can derive from (9) and (10) that the frequency of mismatches (non-zero values $\varepsilon_i(j)$) is equal to β^i, as stated by Theorem 1. ❑

If $\beta \in (0,1)$ is the quadratic unit satisfying $\beta^2 - m\beta = -1$, then the considerations are even simpler, because expression (5) does not depend on the parity of i. We state the result as the following theorem.

Theorem 3

Let $\beta \in (0,1)$ satisfy $\beta^2 - m\beta = -1$ and let $(G_i)_{i=0}^{\infty}$ be defined by (4). For $i \in \mathbb{N}$, put

$$V_i = \left\{k G_{i+1} - (\lfloor k\beta \rfloor + 1) G_i \middle| k \in \mathbb{Z}\right\}.$$

Then for $j \in \mathbb{Z}$ we have

$$\lfloor \beta(j + G_i) \rfloor = \lfloor j\beta \rfloor + G_{i-1} + \varepsilon_i(j),$$

where

$$\varepsilon_i(j) = \begin{cases} 0 & \text{if } j \notin V_i, \\ 1 & \text{otherwise.} \end{cases}$$

The density of the set U_i of mismatches is equal to β^i.

Proof. The proof follows the same lines as proofs of Theorems 1 and 2. ❑

4 Conclusions

One-dimensional cut-and-project sets can be constructed from $\mathbb{Z}^2$ for every choice of straight lines V_1, V_2, if they have irrational slopes. However, in our proof of the self-matching properties of the Beatty sequences we strongly use the algebraic ring structure of the set $\mathbb{Z}[\beta']$ and its scaling invariance with the factor β', namely $\beta'\mathbb{Z}[\beta] = \mathbb{Z}[\beta']$. For this, β' must necessarily be a quadratic unit.

However, it is plausible that, even for other irrationals α, some self-matching property is displayed by the graph $\lfloor j\alpha \rfloor$ against j. To show that, other methods would be necessary.

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