# On a Quantum Waveguide with a Small $\mathcal{PT}$-symmetric Perturbation

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*We consider a quantum waveguide with a small $\mathcal{PT}$-symmetric perturbation described by a potential. We study the spectrum of such a system and show that the perturbation can produce eigenvalues near the threshold of the continuous spectrum.*

*Keywords: waveguide, $\mathcal{PT}$-symmetric potential, spectrum*

In this paper we consider an example of a quantum waveguide with a small $\mathcal{PT}$-symmetric perturbation. The perturbed system is weakly non-self-adjoint and we employ general results of [1, 2] to study the problem. The main aim is to show that the technique suggested in the cited works can be used effectively in the perturbation theory for $\mathcal{PT}$-symmetric operators.

Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$,

$$\Pi = \{x : -\pi/2 < x_2 < \pi/2\}$$

be an infinite straight strip. By $V = V(x)$ we denote a real-valued function defined on $\overline{\Pi}$ having bounded support and belonging to $L_\infty(\Pi)$. We assume that it satisfies the following assumption,

$$V(-x_1, x_2) = -V(x_1, x_2), \quad x \in \Pi. \tag{1}$$

The main object of our study is the operator

$$\mathcal{H}_\varepsilon := -\Delta + \mathrm{i}\,\varepsilon V \tag{2}$$

on $\Pi$ subject to the Dirichlet boundary condition. We define it rigorously as an unbounded operator in $L_2(\Pi)$ with the domain $W_{2,0}^2(\Pi)$, where the latter is a subspace of $W_2^2(\Pi)$ of the functions vanishing on $\partial\Pi$. The symbol $\varepsilon$ indicates a small positive parameter.

The Dirichlet Laplacian on $\Pi$ is a closed operator in $L_2(\Pi)$ and the multiplication operator by $V$ is relatively bounded. Because of this the operator $\mathcal{H}_\varepsilon$ is closed. It can also be shown that it is $m$-sectorial. The main property of the operator $\mathcal{H}_\varepsilon$ is $\mathcal{PT}$-symmetricity expressed by the identity

$$\mathcal{H}_\varepsilon^* = \Pi \mathcal{H}_\varepsilon \Pi^{-1},$$

where $(\Pi u)(x) = u(-x_1, x_2)$.

Our aim is to study the spectrum of the operator $\mathcal{H}_\varepsilon$. We will focus our attention on the continuous, residual and point spectrum of this operator. We define these subsets of the spectrum in accordance with [3]. Namely, the continuous spectrum is introduced in terms of singular sequence, the point spectrum is the set of all eigenvalues, and the residual spectrum is the complement of the continuous and point spectrum with reference to the whole spectrum.

Our first result describes the continuous and residual spectrum of $H_\alpha$.

**Theorem 1**

*The residual spectrum of $\mathcal{H}_\varepsilon$ is empty and the continuous one coincides with $[1, +\infty)$.*

The proof is the same as the proof of similar results in [4]; we therefore do not give the proof here.

It is well-known that the spectrum of operator $\mathcal{H}_0$ is purely continuous and coincides with $[1, +\infty)$. The small perturbation $\varepsilon V$ can generate an eigenvalue converging to the threshold of the continuous spectrum. Our second theorem deals with the existence and asymptotic behaviour of such an eigenvalue. Before formulating it, we introduce auxiliary notations.

We denote

$$\phi_j(x_2) := \frac{\sqrt{2}}{\sqrt{\pi}} \sin jx_2,$$

$$v_j(x_1) := \int_0^\pi V(x)\,\phi_1(x_2)\phi_j(x_2)\,\mathrm{d}x_2,$$

$$u_1(x_1) := -\frac{1}{2}\int_{\mathbb{R}} |x_1 - t_1|\, v_1(t_1)\,\mathrm{d}t_1,$$

$$u_j(x_1) := \frac{1}{\sqrt{j^2 - 1}} \int_{\mathbb{R}} e^{-\sqrt{j^2-1}|x_1 - t_1|}\, v_j(t_1)\,\mathrm{d}t_1.$$

Employing these functions, we define

$$\widetilde{V}(x) := \sum_{j=2}^\infty v_j(x_1)\,\phi_j(x_2) = (V(x) - v_1(x_1))\,\phi_1(x_2),$$

$$\widetilde{u}(x) := \sum_{j=2}^\infty u_j(x_1)\,\phi_j(x_2).$$

The function $\widetilde{u}$ is well-defined and belongs to $W_2^2(\Pi)$. It can be shown that it is an exponentially decaying solution to the problem

$$\begin{aligned} -\Delta\,\widetilde{u} &= \widetilde{u} + \widetilde{V}, & x \in \Pi,\\ \widetilde{u} &= 0, & x \in \partial\Pi. \end{aligned} \tag{3}$$

Finally, we introduce a number $K$ by the formula

$$K := \|u_1'\|^2_{L_2(\mathbb{R})} - (\widetilde{V}, \widetilde{u})_{L_2(\Pi)}.$$

We will show below that the first norm in this formula is well-defined.

**Theorem 2**

*If $K > 0$, there exists the unique eigenvalue of the operator $\mathcal{H}_\varepsilon$, converging to the threshold of the continuous spectrum. This eigenvalue is simple, real and satisfies the asymptotic formula*

$$\lambda_\varepsilon = 1 - \frac{\varepsilon^4 K^2}{4} + \mathcal{O}(\varepsilon^5), \quad \varepsilon \to +0. \tag{4}$$

*If $K < 0$, the operator $\mathcal{H}_\varepsilon$ has no eigenvalues converging to the threshold of the continuous spectrum as $\varepsilon \to +0$. In particular, if*

$$V(x) = v_1(x_1) \tag{5}$$

*the number $K$ is positive, and if*

$$v_1 = 0 \tag{6}$$

*the number $K$ is negative.*

*Proof*. We introduce the numbers

$$k_1 := \frac{i}{2}\int_\Pi V(x)\,\phi_1^2(x_2)\,dx = \int_{\mathbb{R}} v_1(x_1)\,dx_1,$$

$$k_2 := -\frac{1}{2}\int_\Pi (u_1 v_1 \phi_1^2 + \widetilde{V}\widetilde{u})\,dx = -\frac{1}{2}\left(\int_{\mathbb{R}} u_1 v_1\,dx_1 + \int_\Pi \widetilde{V}\widetilde{u}\,dx\right).$$

It follows from [2, Th. 1] that if

$$\operatorname{Re} k_1 > 0, \text{ or } \operatorname{Re} k_1 = 0,\ \operatorname{Re} k_2 > 0, \tag{7}$$

there exists the unique eigenvalue of $\mathcal{H}_\varepsilon$ converging to the threshold of the continuous spectrum, and the asymptotics of this eigenvalue reads as follows

$$\lambda_\varepsilon = 1 - k_\varepsilon^2, \quad k_\varepsilon = \varepsilon k_1 + \varepsilon^2 k_2 + \mathcal{O}(\varepsilon^3), \quad \varepsilon \to +0. \tag{8}$$

It also follows from [2, Th. 1] that if

$$\operatorname{Re} k_1 < 0, \text{ or } \operatorname{Re} k_1 = 0,\ \operatorname{Re} k_2 < 0 \tag{9}$$

the operator $\mathcal{H}_\varepsilon$ has no eigenvalues converging to the threshold as $\varepsilon \to +0$. Thus, it is sufficient to calculate the numbers $k_1$, $k_2$.

The identity (1) implies that $v_1$ is an odd function, and hence

$$k_1 = 0. \tag{10}$$

Therefore, it is sufficient to calculate $k_2$ and check its sign. The mean value of $v_1$ being zero, the function $u_1$ is constant as $|x_1|$ is large enough. This allows us to write

$$\int_{\mathbb{R}} v_1 u_1\,dx_1 = -\int_{\mathbb{R}} u_1'' u_1\,dx_1 = \int_{\mathbb{R}} (u_1')^2\,dx_1. \tag{11}$$

Hence,

$$k_2 = \frac{K}{2}.$$

We substitute this formula and (10) into (8), and by (7), (9) we conclude that if $K > 0$, there exists the unique eigenvalue of $\mathcal{H}_\varepsilon$ satisfying (4). If $K < 0$, the operator has no eigenvalues converging to the threshold of the continuous spectrum. Using (1), one can check easily that $\overline{\lambda}_\varepsilon$ is an eigenvalue of $\mathcal{H}_\varepsilon$ as well. It converges to the threshold and by the uniqueness of such an eigenvalue we conclude that this eigenvalue is real.

Let us prove that the conditions (5), (6) are sufficient for the eigenvalue to be present or absent. Assume first (5). In this case $\widetilde{V} = 0$, $\widetilde{u} = 0$ and

$$K = \frac{1}{2}\|u_1'\|^2_{L_2(\mathbb{R})} > 0 \tag{12}$$

If the relation (6) holds true, the function $u_1$ is identically zero, and

$$K := -(\widetilde{V}, \widetilde{u})_{L_2(\Pi)}$$

Employing (3), by analogy with (11) in the same way we check

$$\int_\Pi \widetilde{V}\widetilde{u}\,dx = \|\nabla\widetilde{u}\|^2_{L_2(\Pi)} - \|\widetilde{u}\|^2_{L_2(\Pi)}. \tag{13}$$

It follows from the definition of the function $\widetilde{u}(x_1,\cdot)$ that

$$\left\|\frac{\partial\widetilde{u}}{\partial x_2}(x_1,\cdot)\right\|^2_{L_2(0,\pi)} \geq 4\|\widetilde{u}\|^2_{L_2(0,\pi)},$$

and hence

$$\|\nabla\widetilde{u}\|^2_{L_2(\Pi)} \geq 4\|\widetilde{u}\|^2_{L_2(\Pi)}.$$

In view of (12), (13) and this estimate we conclude that

$$K \leq -3\|\widetilde{u}\|^2_{L_2(\Pi)} < 0.$$

In conclusion, we observe that the results of [1, 2] allow one also to study also the existence of the eigenvalues emerging from the higher thresholds in the continuous spectrum that are $j^2$. It was shown in [2], that if it exists, this eigenvalue is unique. As in Theorem **2**, this fact implies that the eigenvalue is real and therefore in this case we are dealing with embedded eigenvalues.


## Acknowledgments

This work has been supported in parts by RFBR (05-01-97912) and by the Czech Academy of Sciences and Ministry of Education, Youth and Sports (LC06002). The author has also been supported by a *Marie Curie International Fellowship* within the 6th European Community Framework (MIF1-CT-2005-006254) and gratefully acknowledges support from the Deligne 2004 Balzan Prize in Mathematics and from the the grant of Republic Bashkortostan for young scientists and young scientific collectives.


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