Acta Polytechnica **64**(5):414–419, 2024
https://doi.org/10.14311/AP.2024.64.0414

A NOVEL APPROACH TO NONLINEAR FRACTIONAL VOLTERRA INTEGRAL EQUATIONS

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Abstract. Nonlinear Fractional Volterra integral equations (FVIEs) of the first kind present challenges due to their intricate nature, combining fractional calculus and integral equations. In this research paper, we introduce a novel method for solving such equations using Leibniz integral rules. Our study focuses on a thorough analysis and application of the proposed algorithm to solve fractional Volterra integral equations. By using Leibniz integral rules, we offer a fresh perspective on handling these equations, shedding light on their fundamental properties and behaviours. As a result of this study, we anticipate contributing distinctively to the broader development of analytical tools and techniques. By bridging the gap between fractional calculus and integral equations, our approach not only offers a valuable computational methodology but also paves the way for new insights into the application domains in which such equations arise.

Keywords: Integral equations, fractional calculus, Leibniz integral rule, Mitteg-Leffler function, Caputo fractional operator, Riemann-Liouville fractional operator.

1. Introduction

Fractional calculus is a branch of mathematical analysis that generalises the concept of differentiation and integration to non-integer orders. Unlike traditional calculus, which deals exclusively with integer-order derivatives and integrals, fractional calculus allows for operations involving fractional powers of the differentiation operator. It finds applications in various scientific fields, such as physics, engineering, and biology, where phenomena exhibit fractal or non-local behaviors that cannot be adequately described by classical methods. The study of fractional calculus enables a deeper understanding of complex systems and provides powerful tools for modeling and analyzing intricate dynamical processes with fractional dimensions or memory effects. See for instance [1–4].

Many researchers have studied different methods for solving integral equations of various types: linear, nonlinear, homogeneous, heterogeneous, Volterra, Friedholm, first kind and second kind. Here are some of these studies, in 2010 Mohammed Belmekki and Mouffak Benchohra established sufficient conditions for the existence and uniqueness of solutions for some nondensely defined semilinear functional differential equations involving the Riemann–Liouville derivative [1], in 2015 Salman Jahanshahi, Esmail Babolian, Delfim Torres, and Alireza Vahidi introduced a new method for numerically solving Abel integral equations of the first kind [5], in 2020 Giuseppe Pellicane, Lloyd Lee, and Carlo Caccamo briefly reviewed the application of integral equation theories (IETs) of the fluid state to predict fluid phase equilibria of simple fluids [6], in 2024 Mohammed Abdulshareef Hussein, Hassan Kamil Jassim, and Ali Kareem Jassim introduced the Hussein-Jassim method (HJ-method) for solving Volterra integral equations of the second kind (VIESKs) [7], in 2024 Hamid Mottaghi Golshan proposed a numerical iterative method to solve multidimensional integral equations based on Picard iteration and Newton–Cotes rules in a cubic domain [8]. In addition to other studies [9–16].

The fractional Leibniz integral rules form a key framework in the field of fractional calculus, notably for the Riemann-Liouville and Caputo fractional derivatives, demonstrating their importance in a variety of applications. These rules are an extension of the traditional Leibniz differentiation rule, modified to include fractional orders. Specifically, they allow for the differentiation of intricate compositions and products using fractional derivatives, giving a methodical way to unraveling the complexity of these derivatives. In the context of the Riemann-Liouville fractional derivative, the fractional Leibniz rules are critical in assessing mixed-order derivatives of complex functions, shedding light on their complicated properties [17–19].

This paper aims to introduce a new analytic method for obtaining the exact solution of the following nonlinear fractional Volterra integral equation of the first kind using fractional Leibniz integral rules:

$$f(t) = \int_a^t (t-s)^\gamma g(s) N(\psi(s)) ds, \quad t > a, \tag{1}$$

where $f(t)$ and $g(s)$ are given, $\psi(s)$ is an unknown function, N is a nonlinear term, and $\gamma \geq 0$ is the order of the integral equation. The main idea here is to develop an analytical approach to solve this type of equation using concepts of fractional calculus and to apply Fractional Leibniz integral rules to arrive at exact solutions. This research paper may contribute to a better understanding of how to solve Volterra integral equations of the first kind and may have practical applications in various fields, such as mthematics, science, engineering, and others.

Following is an outline of the paper. In the second section of this paper, we present several mathematical concepts related to our study, in the third section, we discuss the algorithm and analysis of the new method, in the fourth section we apply this method to several illustrative examples to ensure the effectiveness of the method, and in the final section, we review the results and recommendations.

2. Mathematical preliminaries

Definition 1 (The gamma function [20]). *The gamma function, denoted by $\Gamma(\gamma)$ is a mathematical function that generalises the concept of factorial to non-integer values. It is defined as follows:*

$$\Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t} dt, \quad Re(\gamma) > 0. \tag{2}$$

These are the two important properties of the gamma function [20]:

1. $\Gamma(\gamma+1) = \gamma\Gamma(\gamma), \quad \gamma > 0,$ (3)
2. $\Gamma(\gamma+1) = \gamma!,$
   γ *is a non negative integer number.* (4)

Definition 2 (The Mittag-Leffler function [21]). *The Mittag-Leffler function of one parameter is defined by the following power series:*

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma+1)}, \quad Re(\gamma) > 0, \tag{5}$$

and of two parameters is defined by:

$$E_{\gamma,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n+\beta)}, \quad Re(\gamma) > 0, \quad Re(\beta) > 0. \tag{6}$$

Theorem 1 [22]. *Let $E_\gamma(z)$ and $E_{\gamma,\beta}(z)$ are the Mittag-Leffler functions of one and two parameters, respectively. Then:*

1. $E_{\gamma,1}(z) = E_\gamma(z),$ (7)
2. $E_{1,1}(z) = e^z,$ (8)
3. $E_{2,2}(-z^2) = \frac{\sin z}{z},$ (9)
4. $E_{2,1}(-z^2) = \cos z.$ (10)

Definition 3 (The Riemann-Liouville integral [22, 23]). *The Riemann-Liouville integral of fractional order $\gamma > 0$ of a $f \in C[a,b]$ is defined as follows:*

$${}_aI_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} f(\tau) d\tau. \tag{11}$$

Definition 4 (The Riemann-Liouville derivative [22, 23]). *The Riemann-Liouville derivative of fractional order $n-1 < \gamma \leq n$, $n \in \mathbb{N}$ of a $f \in C[a,b]$ is defined as follows:*

$$\begin{aligned} {}^{RL}_{a}D_t^\gamma f(t) &= D_t^n \left({}_aI_t^{n-\gamma} f(t)\right) \\ &= \frac{1}{\Gamma(n-\gamma)} D_t^n \int_a^t (t-\tau)^{n-\gamma-1} f(\tau) d\tau. \end{aligned} \tag{12}$$

Theorem 2 [23]. *Let $\gamma, \beta, \mu, \lambda \in \mathbb{C}$, $Re(\gamma) > 0$, $Re(\beta) > -1$, $n-1 < \gamma \leq n$ and $n \in \mathbb{N}$. Then:*

1. ${}^{RL}D_t^n f(t) = D_t^n f(t),$ (13)
2. ${}^{RL}D_t^\gamma t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} t^{\beta-\gamma},$ (14)
3. ${}^{RL}D_t^\gamma t^\beta E_{\mu,\beta+1}(\lambda t^\mu) = t^{\beta-\gamma} E_{\mu,\beta-\gamma+1}(\lambda t^\mu).$ (15)

Proposition (The Leibniz Integral Rule [24]). *Let $F(x,t)$ be a function of two variables, x and t. If the function $F(x,t)$ and its partial derivative $\frac{\partial F}{\partial x}$ are continuous in a region that includes the interval $[a,b]$ for x, and $a(t)$ and $b(t)$ are continuously differentiable functions of t, then the derivative of the integral is given by:*

$$\begin{aligned} \frac{d}{dt} \int_{a(t)}^{b(t)} F(x,t)\, dx = F(b(t),t) \cdot \frac{db}{dt} - F(a(t),t) \cdot \frac{da}{dt} \\ + \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t} dx. \end{aligned} \tag{16}$$

Theorem 3 [25]. *Let $\gamma > 0$, $t > a$ and $K \in C^2[a,b]$. Then Leibniz integral rule of higher order derivative and fractional Leibniz integral rules in the sense of Riemann-Liouville and Caputo derivatives respectively, are given by:*

1. $$\begin{aligned} D_t^n \int_a^t K(t,s) ds = \sum_{i=1}^{n} D_t^{i-1} \lim_{s\to t} D_t^{n-i} K(t,s) \\ + \int_a^t D_t^n K(t,s) ds, \end{aligned} \tag{17}$$
2. $$\begin{aligned} {}^{RL}_{a}D_t^\gamma \int_a^t K(t,s) ds = \sum_{i=1}^{n} D_t^{i-1} \lim_{s\to t} {}^{RL}_{s}D_t^{\gamma-i} K(t,s) \\ + \int_a^t {}^{RL}_{s}D_t^\gamma K(t,s) ds. \end{aligned} \tag{18}$$

3. The approach analysis

In this section of the research paper, we will provide a detailed analysis of the algorithm of the new method in the context of fractional Volterra integral equations. This method relies on Leibniz integral rules and the algorithm employed in this study constitutes an innovative approach to addressing the challenges posed by fractional Volterra integral equations. This section focuses on the analysis and use of this algorithm for solving fractional Volterra integral equations. Through this approach, we strive to deepen our understanding of the fundamental properties of fractional Volterra integral equations. Furthermore, we will contribute distinctly to the development of analytical tools and techniques available to the scientific research community.

Consider the following Fractional Volterra Integral Equation of first kind such as $g(t) \neq 0$:

$$f(t) = \int_a^t (t-s)^\gamma g(s) N(\psi(s)) ds, \qquad (19)$$

where $n-1 \leq \gamma \leq n$ and $n \in \mathbb{Z}^+$. Now, we will find the exact solution to Equation (19) using the fractional derivative of Riemann-Liouville. By taking the fractional derivative ${}_a^{RL}D_t^{\gamma+1}$ to both sides of Equation (19), we obtain:

$${}_a^{RL}D_t^{\gamma+1} f(t) = {}_a^{RL}D_t^{\gamma+1} \int_a^t (t-s)^\gamma g(s) N(\psi(s)) ds,$$

By using Theorem 3, we get:

$$\begin{aligned}
&{}_a^{RL}D_t^{\gamma+1} f(t) = \\
&= D_t\Bigg(\sum_{i=1}^{n} D_t^{i-1} \lim_{s\to t} {}_s^{RL}D_t^{\gamma-i}(t-s)^\gamma g(s) N(\psi(s)) \\
&\quad + \int_a^t {}_s^{RL}D_t^{\gamma}(t-s)^\gamma g(s) N(\psi(s)) ds\Bigg), \\
&= D_t\Bigg(\sum_{i=1}^{n} D_t^{i-1} \lim_{s\to t} \frac{\Gamma(\gamma+1)}{\Gamma(i+1)}(t-s)^i g(s) N(\psi(s)) \\
&\quad + \int_a^t \Gamma(\gamma+1) g(s) N(\psi(s)) ds\Bigg), \\
&= D_t\left(\int_a^t \Gamma(\gamma+1) g(s) N(\psi(s)) ds\right).
\end{aligned}$$

By definition of the Leibniz integral rule:

$${}_a^{RL}D_t^{\gamma+1} f(t) = \Gamma(\gamma+1) g(t) N(\psi(t)).$$

Assuming that $g(t) \neq 0$, we get:

$$N(\psi(t)) = \frac{{}_a^{RL}D_t^{\gamma+1} f(t)}{\Gamma(\gamma+1) g(t)}.$$

By the concept of the inverse of the function:

$$\psi(t) = N^{-1}\left(\frac{{}_a^{RL}D_t^{\gamma+1} f(t)}{\Gamma(\gamma+1) g(t)}\right). \qquad (20)$$

Thus, the exact solution of Equation (19) when $\gamma = n$ is a non-negative integer number is:

$$\psi(t) = N^{-1}\left(\frac{D_t^{n+1} f(t)}{n! g(t)}\right). \qquad (21)$$

4. Illustrative examples

In this section, we will use the new method we have developed to solve various Volterra integral equations. Our goal is to obtain precise solutions for these equations and assess the accuracy and efficacy of our method in achieving these solutions. Throughout this section, we will provide detailed explanations of the steps we take, the equations we solve, and the results we achieve. This section will showcase our novel method's application to various Volterra integral equations. By obtaining exact solutions and evaluating their accuracy and computational efficiency, we aim to establish the credibility and utility of our method as a valuable tool for solving these types of equations in diverse scenarios.

Example 1. *Consider the following Abel's integral equation [26]:*

$$\pi + t = \int_0^t \frac{e^{\psi(s)}}{\sqrt{t-s}} ds, \qquad (22)$$

with the algorithm of the new method, we get:

$$f(t) = \pi + t, \quad g(t) = 1, \quad \gamma = \frac{-1}{2}.$$

Now:

$$\begin{aligned}
e^{\psi(s)} &= \frac{{}^{RL}D_t^{\frac{1}{2}}(\pi + t)}{\Gamma(1/2)} \\
&= 2\left(\frac{2t+\pi}{\pi\sqrt{t}}\right).
\end{aligned}$$

Thus, the exact solution of Equation (22) *is given by:*

$$\psi(t) = \ln\left(2\frac{2t+\pi}{\pi\sqrt{t}}\right).$$

Example 2. *Assume the following Volterra integral equation [27]:*

$$\frac{9}{7}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)t^{7/6}}{\sqrt{\pi}} = \int_0^t \frac{\ln(\psi(s))}{\sqrt[3]{t-s}} ds, \qquad (23)$$

from Equation (20)*, we obtain:*

$$f(t) = \frac{9}{7}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)t^{7/6}}{\sqrt{\pi}}, g(t) = 1, \gamma = \frac{-1}{3}.$$

Now:

$$\begin{aligned}
\ln(\psi(t)) &= \frac{1}{\Gamma(2/3)}\, {}^{RL}D_t^{\frac{2}{3}}\left(\frac{9}{7}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)t^{7/6}}{\sqrt{\pi}}\right) \\
&= \sqrt{t}.
\end{aligned}$$

Thus, we have the exact solution of Equation (23)*:*

$$\psi(t) = e^{\sqrt{t}}.$$

Example 3. *Suppose that the following FVIE [27]:*

$$\frac{6\,t^{5/6}\,(11+6\,t)}{55} = \int_0^t \frac{\arcsin(\psi(s))}{\sqrt[6]{t-s}} ds, \qquad (24)$$

by using our algorithm, we obtain:

$$f = \frac{6\,t^{5/6}\,(11+6\,t)}{55}, g(t) = 1, \gamma = \frac{-1}{6}.$$

Now:

$$\arcsin(\psi(t)) = {}^{RL}D_t^{3/2} \frac{6\, t^{5/6}\,(11+6\,t)}{55} = t+1.$$

Hence, we obtain the exact solution of Equation (24):

$$\psi(t) = \sin(t+1).$$

Example 4. *Consider the following FVIE of first kind [27]:*

$$\frac{\sqrt{\pi}\,\Gamma(\frac{5}{3})}{38\,\Gamma(\frac{19}{6})}(10t^{\frac{19}{6}} + 19t^{\frac{13}{6}}) = \int_0^t \sqrt{t-s}(1+s)u^2(s)ds, \quad (25)$$

by Equation (20), we get:

$$f(t) = \frac{\sqrt{\pi}\,\Gamma(\frac{5}{3})}{38\,\Gamma(\frac{19}{6})}(10t^{19/6} + 19t^{13/6}), \quad g(t) = 1+t, \quad \gamma = \frac{1}{2}.$$

Now:

$$u^2(t) = \frac{\sqrt{\pi}\,\Gamma(\frac{5}{3})}{38\,\Gamma(\frac{19}{6})\Gamma(\frac{3}{2})(1+t)}\ {}^{RL}D_t^{3/2}\Big(10t^{\frac{19}{6}} + 19t^{\frac{13}{6}}\Big) = \frac{t^{5/3}+t^{2/3}}{1+t} = t^{2/3}.$$

Therefore, the following solution results from Equation (25):

$$\psi(t) = \pm\sqrt[3]{t}.$$

Example 5. *Let the following FVIE [26]:*

$$\frac{1}{2}\sqrt{\pi x^3}E_{1,\frac{3}{2}}(\frac{t}{2}) = \int_0^t \sqrt{(t-s)\psi(s)}ds, \qquad (26)$$

With Equation (20), we get:

$$f(t) = \frac{1}{2}\sqrt{\pi x^3}E_{1,\frac{3}{2}}(\frac{t}{2}), \quad g(t) = 1, \quad \gamma = \frac{1}{2}.$$

Now:

$$\sqrt{\psi(t)} = \frac{1}{2}\ {}^{RL}D_t^{3/2}\left(\sqrt{\pi x^3}E_{1,3/2}(\frac{1}{2}t)\right) = E_{1,1}(\frac{1}{2}t) = e^{\frac{t}{2}}.$$

Thus, we get an accurate solution of Equation (26):

$$\psi(t) = e^t.$$

Example 6. *Assume the following singular FVIE [28]*

$$\frac{4x^{3/2}}{3} - \frac{32\,x^{7/2}}{35} = \int_0^t \frac{\psi(s)}{\sqrt{t-s}}\,ds \qquad (27)$$

By the previous algorithm, we get:

$$f(t) = \frac{4x^{3/2}}{3} - \frac{32\,x^{7/2}}{35}, \quad g(t) = 1, \quad \gamma = \frac{-1}{2}.$$

Now:

$$\psi(t) = \frac{1}{\Gamma(1/2)}\ {}^{RL}D_t^{3/2}\left(\frac{4x^{3/2}}{3} - \frac{32\,x^{7/2}}{35}\right).$$

Thus, the solution of Equation (27) becomes:

$$\psi(t) = t - t^3.$$

Example 7. *Let the following Abel integral equation [28]:*

$$\frac{2\sqrt{t}\left(-48\,t^3 - 56\,t^2 + 105\right)}{105} = \int_0^t \frac{\psi(s)}{\sqrt{t-s}}\,ds. \quad (28)$$

By Equation (20), we get:

$$f(t) = \frac{2\sqrt{t}\left(-48\,t^3 - 56\,t^2 + 105\right)}{105}, \quad g(t) = 1, \quad \gamma = \frac{-1}{2}.$$

Now:

$$\psi(t) = \frac{1}{\Gamma(\frac{1}{2})}\ {}^{RL}D_t^{1/2}\left(\frac{2\sqrt{t}\left(-48\,t^3 - 56\,t^2 + 105\right)}{105}\right).$$

Thus, we have

$$\psi(t) = 1 - t^2 - t^3.$$

Remark. The exact solutions in Examples 1 and 5 obtained by the new method are completely identical to the precise solutions in the reference [26]. In Examples 1, 2, and 3, we obtained precise solutions to the integral equations using the new method, identical to the exact solutions in the reference [27]. Examples 6 and 7 have numerical solutions that converge to the exact solutions we obtained using the new method; see the reference [28]. From the above, it becomes clear that the method that was used in this study is an effective and efficient method for solving fractional integral equations of the first kind. This method is characterised by its simple and concise algorithm.

5. Results and conclusion

In conclusion, this study introduces a new method using the Leibniz integration rule to solve exactly nonlinear Volterra integral equations of the first kind with fractional orders. The method's effectiveness is demonstrated through illustrative examples provided in the study. Applying this approach to several equations of fractional orders consistently yielded satisfactory results. Researchers are encouraged to further develop and refine this method for solving a wide range of integral and differential equations.

Data availability No underlying data were collected or produced in this study.

Conflicts of interest We, the authors, declare no conflict of interest.

Funding The authors did not receive any financial support.

Acknowledgements

The authors would like to express their sincere gratitude to all those who contributed to this work.

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