Correlation Functions for Lattice Integrable Models

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In this lectures I consider the problem of calculating the correlation functions for XXZ spin chain. First, I explain in details the free fermion case. Then I show that for generic coupling constant the fermionic operators acting on the space of quasi-local fields can be introduced. In the basis generated by these fermionic operators the correlation functions are given by determinants as in the free fermion case.

Keywords: Quantum integrable models, correlation functions, exactly solvable models of statistical physics.

1 Lecture 1.

These lectures are based on a series of papers written in collaboration with H. Boos, M. Jimbo, T. Miwa, Y. Takeyama [1, 2, 3, 4, 5, 6, 7].

Consider the finite chain of length \( N \) with periodical boundary conditions (we assume that \( N \) is even):

\[
H_{N/2}^{(X)} = i \sum_{k=N/2}^{N-1} (\psi_{k}^\dagger \psi_{k+1} + \psi_{k} \psi_{k+1}^\dagger).
\]

Consider a finite chain of length \( N \) with periodical boundary conditions (we assume that \( N \) is even):

\[
H^{(N/2)}_{XX} = \frac{1}{\sqrt{N}} \sum_{k=N/2}^{N-1} \psi_{k}^\dagger \bigg[ \frac{1 + \xi_2^{N/2}}{1 - \xi_2^{N/2}} \bigg]^{1/2} \psi_{k}.
\]

I apologize for the strange parametrization of the momentum, but it will be useful in the generic case. We have

\[
\langle [\psi^+(\xi_1), \psi^-(\xi_2)] \rangle_{+} = 0,
\]

\[
[\psi^+(\xi_1), \psi^-(\xi_2)]_{+} = \left( \frac{1 + \xi_2^{N/2}}{1 - \xi_2^{N/2}} \right)^{N/2} - 1.
\]

So, introducing \( \tau_j \) as solutions to

\[
\frac{1 - r_j^2}{1 + r_j^2} = e^{-N \tau_j}, \quad j = -N/2, \ldots, N/2 - 1,
\]

we have:

\[
\langle [\psi^+(\tau_j), \psi^-(\tau_j)] \rangle_{+} = \delta_{l,j}.
\]

The Hamiltonian is easily expressed in terms of the Fourier transform:

\[
H^{(N/2)}_{XX} = \sum_{k=N/2}^{N-1} \psi^+(\tau_j) \psi^-(\tau_j) \sin \left( \frac{2\pi k}{N} \right).
\]
As usual in fermionic models, there are negative energies which are taken care of by the Dirac trick: rewrite the Hamiltonian as
\[
H_{\mathcal{X}X}^{(1)} = \sum_{k=0}^{N/2} \psi^+(\tau_j) \psi^-(\tau_j) \sin \left( \frac{2\pi j}{N} \right) + \text{Const}
\]
which means that the vacuum must satisfy
\[
\psi^+(\tau_j) |\text{vac}\rangle = 0, \quad j = -N/2, \ldots, -1
\]
\[
\psi^-(\tau_j) |\text{vac}\rangle = 0, \quad j = 0, \ldots, N/2 - 1.
\]

So, if we start with a ferromagnetic vacuum $|0\rangle$ in which all spins are down, the real vacuum $|\text{vac}\rangle$ is obtained by filling the Dirac sea:
\[
|\text{vac}\rangle = \prod_{j=-N/2}^{-1} \psi^+(\tau_j) |0\rangle.
\]

I have considered this simple example in order to explain that even in the free fermion case the vacuum is a rather complicated state if we present it in terms of original spin variables.

Now we have to take the limit $N \rightarrow \infty$. For the Fourier transform we have
\[
\psi^\pm = \int \frac{d\xi}{\sqrt{2\pi}} \frac{1 - \xi^2}{1 + \xi^2} \hat{\psi}^\pm(\xi) \sqrt{2\xi^2 - 1},
\]
and the vacuum satisfies:
\[
\psi^+(\xi) |\text{vac}\rangle = 0, \quad \text{phase} \left( \frac{1 - \xi^2}{1 + \xi^2} \right) < 0,
\]
\[
\psi^-(\xi) |\text{vac}\rangle = 0, \quad \text{phase} \left( \frac{1 - \xi^2}{1 + \xi^2} \right) < 0.
\]

Let me give some explanation about the case of generic $q$.

It is well known nowadays that the integrability of the XXZ model is due to its relation to the trigonometric $R$-matrix. The $R$-matrix belongs to the tensor product of the two algebras of the 2 matrices: $R(\xi) \in \text{Mat}(2, \mathbb{C}) \otimes \text{Mat}(2, \mathbb{C})$:
\[
R(\xi) = \frac{1}{\zeta \xi - \xi^{-1} q^{-1}}
\begin{pmatrix}
\zeta - \xi^{-1} q^{-1} & 0 & 0 & 0 \\
0 & \zeta - \xi^{-1} & q^{-1} & 0 \\
0 & q^{-1} & \zeta - \xi^{-1} & 0 \\
0 & 0 & 0 & \zeta \xi - \xi^{-1} q^{-1}
\end{pmatrix}
\]

The $R$-matrix satisfies the Yang-Baxter equation:
\[
R_{12}(\xi_1 \xi_2) R_{13}(\xi_1 \xi_3) R_{23}(\xi_2 \xi_3) = R_{23}(\xi_2 \xi_3) R_{13}(\xi_1 \xi_3) R_{12}(\xi_1 \xi_2).
\]

where as usual I shall denote by $R_{ij}(\xi)$ the $R$-matrix acting in the tensor product of two copies of $\text{Mat}(2, \mathbb{C})$. I shall also use the notion of an $L$-operator. In this particular case there is no difference between these two things, and I shall explain later what I mean by an $L$-operator in general. Here I consider two copies of $\text{Mat}(2, \mathbb{C})$ one of which is called auxiliary (carrying the index $a$), and the other is called quantum (carrying the index $j$). So, $L_{a,j}(\xi)$ is $R(\xi)$ as an element of the tensor product of these two algebras.

The most important object of the theory of integrable models is the transfer-matrix:
\[
t_N(\xi) = \text{tr}_a(\tilde{L}_{a,N/2-1}(\xi) \cdots \tilde{L}_{a,-N/2}(\xi)).
\]
The Yang-Baxter equation implies commutativity:
\[
[t_N(\tilde{\xi}), t_N(\xi)] = 0.
\]

It is well known that the Hamiltonian of a periodic XXZ chain on $N$ sites is contained in this one-parametric family of operators, but since I shall need some knowledge about higher local integrals of motion let me repeat the derivation of this fact.

It is convenient to introduce
\[
\tilde{L}_{a,j}(\xi) = \xi^\frac{q}{2} \tilde{L}_{a,j}(\xi)^{\frac{q}{2}}.
\]

Note that we still have
\[
t_N(\xi) = \text{tr}_a(\tilde{L}_{a,N/2-1}(\xi) \cdots \tilde{L}_{a,-N/2}(\xi))
\]
because $t_N(\xi)$ commutes with the total spin. Introduce further
\[
L'_{a,j}(\xi) = \tilde{L}_{a,j}(\xi)^{P_{a,j}} = 1 + \frac{\zeta^2 - 1}{\zeta^2 - q^2} h_{a,j},
\]
where
\[
h_{a,j} = \left( a^+_a a_j + a^+_a a^-_j + \frac{q + q^{-1}}{4} (a^+_a a^-_j - 1) + \frac{q - q^{-1}}{4} (a^+_a - a^-_j) \right) \frac{\zeta^2}{\xi} - \xi^{-1} q^{-1}.
\]

Note that the operator $h_{a,j}$ (the density of the Hamiltonian) is a projector, so it is easy to see that
\[
L'_{a,j}(\xi) = \exp \left( \frac{1}{q + q^{-1}} \log \frac{q - q^{-1}}{\zeta^2 q - q^{-1}} h_{a,j} \right).
\]

Now we calculate
\[
t_N(\xi) = \tilde{L}^{1}_{N,2N/2-1}(\xi) \tilde{L}^{2}_{N/2-1,N/2-2}(\xi) \cdots \tilde{L}^{N/2+1}_{N/2-1}(\xi)
\times P^{1}_{N,2N/2-1} \cdots P^{N}_{N,2N/2+1} = e^{\epsilon L_b} U,
\]
where $\tilde{L}^{1}_{N,2N/2-1}$ means that the operators from the $-N/2$ ($N/2 - 1$) tensor component are put to the right (left) of the product of the $L$-operators. It is easy to see that $L_b = H^{(N)}_{\text{XXZ}}$, which is the Hamiltonian of the periodic XXZ chain. Moreover, from the Campbell-Hausdorf formula one concludes that
\[
I_p = \sum_j d_{p,j}
\]
where \( d_{p,j} \) acts nontrivially from the \( j \)-th to the \((j+p)\)-th site. These operators are called local integrals of motion: they commute with the Hamiltonian, and they are composed of local densities.

Finally, I would like to say what the ground state of the Hamiltonian of the periodic chain looks like. Denote

\[
L_{a,N/2-1}(\xi) \cdots L_{a,-N/2}(\xi) = \begin{pmatrix}
A_N(\xi) & B_N(\xi) \\
C_N(\xi) & D_N(\xi)
\end{pmatrix}.
\]

The eigenvectors of the transfer matrix with total spin 0 (a vacuum is among them) are constructed as

\[
C_N(\tau_1) \cdots C_N(\tau_{N/2}) \bigg| 0 \bigg.
\]

where \( \tau_j \) solve the B equations:

\[
\left( 1 - s_j^2 \right)^N \prod_{k,j} \frac{s_j^2 - q s_k^2}{\phi_j \phi_k} = 1 - s_j^2 \quad j = 1, \ldots, N/2.
\]

A vacuum corresponds to a particular solution to these equations which is a continuous deformation from the free fermion case. So, we see that the vacuum is a complicated object which is produced by the action of two chiral Virasoro algebras: the space of spinless descendants is spun by the vectors.

This is a simple exercise on normal reordering of an exponent of quadratic form. One finds:

\[
\varphi_0 = \frac{(\beta - \beta^{-1})}{2} (\phi + \tilde{\phi})
\]

where \((\phi + \tilde{\phi})\) are two chiral bosonic fields. The descendants are created by the action of two chiral Virasoro algebras:

\[
[\hat{L}_m, \hat{L}_n] = \frac{1}{12} n(n^2 - 1) \delta_{m-n},
\]

\[
[\hat{L}_m, \hat{\varphi}_0] = \frac{1}{12} n(n^2 - 1) \varphi_0.
\]

We have:

\[
\hat{L}_m(\varphi_0) = \hat{L}_m(\varphi_0) = 0, \quad m > 0,
\]

\[
\hat{L}_0(\varphi_0) = \Delta_\alpha \varphi_0,
\]

where \( \Delta_\alpha = (\alpha \beta - \beta^{-1})^2 / 8 \pi \) is the anomalous dimension. The space of spinless descendants is spun by the vectors

\[
\tilde{L}_{-k_1} \cdots \tilde{L}_{-k_p} \cdot L_{-j_1} \cdots L_{-j_l} \varphi_0.
\]

After perturbation, the descendants acquire the vacuum expectation values. The question which A. Zamolodchikov asked me long ago is whether it is possible to write find a covector \( \{ \varphi(L_1, L_2, \ldots, L_j, \tilde{L}_1, \tilde{L}_2, \ldots) \} \) (here \( \{ \} \) has nothing to do with what was previously used, it is the Virasoro left vacuum and it is Virasoro left vacuum \( \{ \varphi(L_{-k_1} \cdots L_{-k_p}) \} \) for \( q \geq 0 \) ) such that the vacuum expectation values are given by the scalar products of this covector with descendants. At that time I was unable to answer this question, but in these lectures I shall explain a construction which is very much in the spirit of this point of view.

So, our goal is to start working in the space of the operators and to introduce the operators acting in this space. Let me give some formal definitions. I have already defined the space \( W_{a,0} \). Similarly I define the spaces \( W_{a,s} \) of operators \( q^{20sN(0)} \) of spin \( s \), and

\[
W_{a,s} = \bigoplus_{s=-\infty}^{\infty} W_{a,s}.
\]

Also, we shall shift \( s \) by integers, so I introduce the space:

\[
W_{a,s} = \bigoplus_{s=-\infty}^{\infty} W_{a,s}.
\]
We shall need an analogue of the left Virasoro vacuum, i.e. a linear functional on $W_{\alpha}$. First, introduce also the linear functional on $\text{End}(C^2)$:

$$\Omega = \frac{1}{\sin \frac{\alpha}{2}} \sum_{\nu} \Omega_{\nu} = \frac{1}{\sin \frac{\alpha}{2}} \sum_{\nu} \Omega_{\nu}$$

(2.1)

where $\Omega_{\nu}$ denote local operators living on a positive axis only.

Still I am not quite happy with this formula, because there is a problem with translational invariance. I would like to avoid using $\Omega_{\nu}$. I introduce the following operators:

$$b_{\nu} \Omega_{\nu,0} = \frac{2i(-1)^{\nu}}{1 - (-1)^{\nu} \xi^{1-\nu}} \sin \frac{\xi}{2}$$

$$c_{\nu} \Omega_{\nu,0} = 2y \xi^{1-\nu} \sin \frac{\xi}{2}$$

The last formula we consider $\Phi_{\nu,j}^{\pm}, \Phi_{\nu,j}^{\mp}$ as components of a row (resp. column) vector,

$$M = (1 + u)(1 - u)^{-1}, \quad (u \Psi_{\nu}^{\pm})_j = \Psi_{\nu,j+1}^{\pm}$$

and $\log(I - \xi^2 M)$ are understood as Taylor series in $u$. $N[\cdot]$ stands for the normal ordering, which applies only to operators acting at the same site. For them we set

$$N[\Phi_{\nu,j}^{\pm}, \Phi_{\nu,j}^{\mp}] = \left\{ \begin{array}{ll}
\Phi_{\nu,j}^{\pm} & (j > 0), \\
-\Phi_{\nu,j}^{\mp} & (j \leq 0).
\end{array} \right.$$
Moreover, by definition
\[ b(\xi) = \sum_{p=1}^{\infty} \frac{1}{(\xi^2 - 1)^p} b_p \xi^{-p-1}, \]
\[ c(\xi) = \sum_{p=1}^{\infty} \frac{1}{(\xi^2 - 1)^p} c_p \xi^{-p-1}, \]
then
\[ b_p(q^{2\alpha S}(0)\mathcal{O}) = 0, \quad p > \text{length}(q^{2\alpha S}(0)\mathcal{O}), \]
\[ c_p(q^{2\alpha S}(0)\mathcal{O}) = 0, \quad p > \text{length}(q^{2\alpha S}(0)\mathcal{O}). \]
This implies that for any \( \mathcal{O} \) only a finite number of terms in the series for \( \alpha^2 \) count. Another corollary is that \( b(\xi)(q^{2\alpha S(k)}) = c(\xi)(q^{2\alpha S(k)}) = 0 \), so we have translational invariance as was announced.

4. Recall that the local integrals of motion have the form:
\[ I_p = \sum d_{p,p}. \]
So, obviously their adjoint action is well defined on \( W_\alpha \). Denote \( I_p(X) = [I_p, X] \). We have:
\[ [I_p, b(\xi)] = [I_p, c(\xi)] = 0. \]
This property is very important for self-consistency of our main formula, because the vacuum expectation value of \( I_p(X) \) must vanish.

For the moment, all this was for the free fermion case. However, the main result of our research is that one can give an algebraic definition of operators \( b(\xi), c(\xi) \) in the case of generic \( p=1 \). They possess the same properties, and the formula for the vacuum expectation value is exactly the same, with the only difference that the function \( \omega(\xi, \alpha) \) should be replaced by:
\[ \omega(\xi, \alpha) = \frac{4(q^\alpha)^\alpha}{(1 + q^\alpha)^2} \left( \frac{q^{-\alpha} - 1}{1 - q^{-\alpha} \xi^2} \right) \sin \left( \frac{\pi}{2}(u - v(u + \alpha)) \right) \]
\[ + \int_{-\pi}^{\pi} \frac{\sin \frac{\pi}{2}(u - v(u + \alpha))}{\sin \frac{\pi}{2} u \cos \frac{\pi}{2}(u + \alpha)} \cos u \, du. \]

I shall define the operators \( b(\xi), c(\xi) \) in the last lecture, but I would like to finish the present lecture by completing the set of operators. Namely, we are able to define the creation operators, \( b^*(\xi), c^*(\xi) \) and an additional operator \( t^*(\xi) \). For the free fermion case one can write explicit formulae for these operators. In the generic case they are constructed similarly to what I shall explain in the next lecture. Their properties are as follows.

1. The block structure of \( b^*(\xi), c^*(\xi), t^*(\xi) \) is as follows:
\[ b^*(\xi): W_{\alpha-1\alpha+1} \rightarrow W_{\alpha\alpha}, \]
\[ c^*(\xi): W_{\alpha+1\alpha-1} \rightarrow W_{\alpha\alpha}, \]
\[ t^*(\xi): W_{\alpha\alpha} \rightarrow W_{\alpha\alpha}. \]

2. These operators have the following commutation relations with \( b(\xi), c(\xi) \):
\[ [b(\xi_1), c^*(\xi_2)]_\alpha = [c(\xi_1), b^*(\xi_2)]_\alpha = [c(\xi_1), t^*(\xi_2)]_\alpha = [b(\xi_1), t^*(\xi_2)]_\alpha = 0, \]
\[ [b(\xi_1), b^*(\xi_2)]_\alpha = \left( \frac{\xi_1}{\xi_2} \right)^\alpha \left( 1 - \frac{\xi_1}{\xi_2} \right), \]
\[ [c(\xi_1), c^*(\xi_2)]_\alpha = \left( \frac{\xi_2}{\xi_1} \right)^\alpha \left( 1 - \frac{\xi_2}{\xi_1} \right). \]

3. As functions of \( \xi \) the operators \( b^*(\xi), c^*(\xi), t^*(\xi) \) are:
\[ b^*(\xi) = \sum_{p=1}^{\infty} (\xi^2 - 1)^p b_p \xi^{-p-1}, \]
\[ c^*(\xi) = \sum_{p=1}^{\infty} (\xi^2 - 1)^p c_p \xi^{-p-1}, \]
\[ t^*(\xi) = \sum_{p=1}^{\infty} (\xi^2 - 1)^p t_p \xi^{-p-1}. \]

4. The operators \( b^*(\xi), c^*(\xi), t^*(\xi) \) acting on \( q^{2\alpha S(k)} \) create the space \( W_\alpha \). The locality is respected due to the properties:
\[ \text{length}(b_p(q^{2\alpha S}(0)\mathcal{O})) \leq \text{length}(q^{2\alpha S}(0)\mathcal{O}) + p, \]
\[ \text{length}(c_p(q^{2\alpha S}(0)\mathcal{O})) \leq \text{length}(q^{2\alpha S}(0)\mathcal{O}) + p, \]
\[ \text{length}(t_p(q^{2\alpha S}(0)\mathcal{O})) \leq \text{length}(q^{2\alpha S}(0)\mathcal{O}) + p. \]

5. The operators \( b^*(\xi), c^*(\xi), t^*(\xi) \) respect \( \alpha^2 \):
\[ \text{tr}^2(b^*(\xi)(X)) = \text{tr}^2(c^*(\xi)(X)) = 0, \quad \text{tr}^2(t^*(\xi)(X)) = \text{tr}^2(X). \quad (2.3) \]

For the vacuum expectation value we have:
\[ \langle \text{vac} | b^*(\xi_1) \cdots b^*(\xi_p) c^*(\xi_1) \cdots c^*(\xi_p) t^*(\xi_1) \cdots t^*(\xi_p) | q^{2\alpha S(0)} \text{vac} \rangle = \]
\[ = \text{det} w(\xi_j \xi_j, \alpha) \]
I think this is a good point to finish this lecture.

3 Lecture 3.

In the previous lecture I claimed that the space of operators \( W_\alpha \) can be organized in such a way that the vacuum expectation values are easy to calculate. Namely, I claimed that there are operators \( b(\xi), c(\xi), b^*(\xi), c^*(\xi), t^*(\xi) \) which can be constructed algebraically, and which provide this organization of \( W_\alpha \). But for the moment I described some of these operators for the case of free fermions only. Now I shall explain the general construction, but I think I shall not be able to do this for all the operators. I shall therefore restrict myself to the operator \( c(\xi) \). If the construction of these operators is clear to you at the end of this lecture I shall be quite happy. Other operators are constructed using similar means.

First, let me prepare our notation for the \( L \)-operators. Consider the quantum affine algebra \( U_q(\hat{\mathfrak{b}}_2) \). The universal \( R \)-matrix of this algebra belongs to the tensor product \( b_n \otimes b_n \), of its two Borel subalgebras. By an \( L \)-operator we mean its image under an algebra map \( b_n \otimes b_n \rightarrow N_1 \otimes N_2 \), where \( N_1, N_2 \) are some algebras. I shall always take \( N_2 \) to be the algebra
Let us consider the finite chain with sites from $k$ to $l$. By $M_{[k,l]}$ I shall denote $\text{Mat}(2,\mathbb{C})^{(l-k+1)}$. As usual, the main object is the monodromy matrix:

$$T_{[k,l]}(\xi) = T_{[1,k]}(\xi) \cdots T_{[k,l]}(\xi),$$

where $\xi$ stands for any auxiliary algebra. However, contrary to the usual situation I want to act on the operators, so, I introduce the adjoint action:

$$T_{[k,l]}(\xi)^{-1} = T_{[k,l]}(\xi) \cdot T_{[1,k]}(\xi)^{-1}.$$

Consider $X \in M_k$ and define

$$\begin{align*}
\mathbb{A}_{[k,l]}(\xi,\alpha) &\to \mathbb{C}_{[k,l]}(\xi,\alpha) \\
\mathbb{D}_{[k,l]}(\xi,\alpha) &\to (X) = T_{[1,k],\alpha}(\xi) \\
(\alpha^{(2l+1)D_A+1} + \alpha^{2}\alpha_A)q^{-2S_{[k,l]}(\xi)}X,
\end{align*}$$

where $S_{[k,l]} = \frac{1}{2} \sum_{j<k} \alpha_j^3$. From the fusion relation we obtain:

$$\begin{align*}
\mathbb{A}_{[k,l]}(\xi,\alpha) &= q^{-2S_{[k,l]}(\xi)}T_{[1,k],\alpha}(\xi)q^{-S_{[1,k],\alpha}}(\xi)(X), \\
\mathbb{D}_{[k,l]}(\xi,\alpha) &= q^{a^{-1}}T_{[1,k],\alpha}(\xi)q^{-S_{[1,k],\alpha}}(\xi)(X),
\end{align*}$$

where $S$ stands for the adjoint action of total spin.

Now I introduce the most important object:

$$\begin{align*}
\mathbb{C}_{[k,l]}(\xi,\alpha)(X) &= T_{[1,k],\alpha}(\xi)C_{[k,l]}(\xi,\alpha) \xi^{-a^{-1}}(X),
\end{align*}$$

which satisfies the crossing symmetry relation:

$$L_{[k,l]}(\xi)^{-1} = \frac{1}{\xi - \xi^{-1}} T_{[k,l]}(\xi).$$
the trace is taken with respect to $W^+$, it converges for $|q^2| < 1$, and continues analytically to the other $\alpha$.

There is one obvious property of $c_{[k,l]}(\xi, \alpha)$:
\[
c_{[k,l]}^{(0)}(\xi, \alpha) q^{-\alpha} X_{[k+1,l]} = q^{-(\alpha-1)} q^{-\alpha} c_{[k,l]}^{(0)}(\xi, \alpha)(X_{[k+1,l]}).
\]

It follows from the definition of the adjoint and $U(1)$-symmetry of the $L$-operator. We call this property the first reduction relation.

In addition there is another property which is a result of non-trivial calculation:
\[
c_{[k,l]}^{(0)}(\xi, \alpha)(X_{[k,l-1]} I_l) = c_{[l,k-1]}^{(0)}(\xi, \alpha)(X_{[k,l-1]} I_l) + \Delta(F'(\xi)),
\]
where $\Delta(f(\xi)) = f(\xi q) - f(\xi q^{-1})$, the explicit expression for $F'(\xi)$ is irrelevant here. So, the reduction relation from the right is satisfied up to the “$q$-exact form”. In classical mathematics the additional term is usually eliminated by integrating over the closed cycle. Here it is similar. Obviously, $c_{[k,l]}^{(0)}(\xi, \alpha)$ is singular at $q^2 = 1, q^2 = -1$. Then the above relation implies that for
\[
c_{[k,l]}(\xi, \alpha) = \frac{\sin q^2}{q^2} \left( c_{[k,l]}^{(0)}(\xi, \alpha) + c_{[k,l]}^{(0)}(\xi, \alpha) \right)
\]
we have
\[
c_{[k,l]}(\xi, \alpha)(X_{[k,l-1]} I_l) = c_{[l,k-1]}(\xi, \alpha)(X_{[k,l-1]} I_l).
\]

This is the second reduction relation. Now we are able to define the operator $e(\xi, \alpha)$ acting from $W_\alpha$ to $W_{\alpha-1}$ taking the inductive limit. Indeed consider $X \in W_\alpha$, denote by $X_{[k,l]}$ its restriction to the interval $[k,l]$. Then
\[
e(\xi, \alpha)(X) = \lim_{k \to \infty, l \to \infty} c_{[k,l]}(\xi, \alpha)(X_{[k,l]})
\]
due to the reduction relation the limit is well-defined since for a large enough interval $[k,l]$ the sequence stabilizes!

The commutation relations are proved similarly. I do not go into details, but using the $R$-matrix one can show that
\[
c_{[k,l]}^{(0)}(\xi_1, \alpha - 1)c_{[k,l]}^{(0)}(\xi_2, \alpha) + c_{[k,l]}^{(0)}(\xi_2, \alpha - 1)c_{[k,l]}^{(0)}(\xi_1, \alpha)
\]
= $\Delta_{\xi_1}$($F(\xi_1, \xi_2)$) + $\Delta_{\xi_2}$($F(\xi_2, \xi_1)$),
\]
so, there is anticommutativity up to the “$q$-exact 2-form”. Again, “integrating over the closed cycle” (passing to $e(\xi)$) one obtains anticommutativity.

Let me finish these lecture with some general remarks. The construction of other operators is not very different from $c$, but for $b^*$ and $c^*$ some additional work has to be done. The commutation relations are not always easy to prove because the $R$-matrices are not always applicable.

I did not say anything about the derivation of our formula for the vacuum expectation values for generic $q$. Actually, it was obtained as a result of a long transformation of the integral formula by Jimbo and Miwa [12]. However, I find quite unsatisfactory that we have such a complicated derivation of such a simple result. I hope to find a more direct proof.

References


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