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CHANCE CONSTRAINED STOCHASTIC PROGRAMMING IN DESIGN OF A FRAME STRUCTURE

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ABSTRACT

A civil engineering problem concerning the optimal design of a loaded frame structure with a random Young's modulus is discussed. The developed multi-criteria optimization model involves ODE-type constraints and also one chance constraint related to the structure’s reliability. A computational scheme for this type of problem is proposed using the finite difference method for the approximation of the ODE constraint and the scenario-based approach for random variable approximation. The chance constraint is handled by two approaches – the analytical approach and penalty reformulation. A posteriori check of satisfying the chance constraint is made, and the upper bounds of the obtainable reliability are computed.

KEYWORDS

Optimal engineering design, Stochastic programming, Chance constrained optimization, Ordinary differential equations

INTRODUCTION

It is often important in engineering to find the best possible design of a studied structure. To ensure the best solution, it is necessary to use a suitable optimization method. Furthermore, it is useful to consider several optimization criteria, e.g. minimizing the cost (which corresponds to the weight of the designed structure) and maximizing its rigidity. Optimal design problems in all engineering areas often lead to multicriteria optimization models constrained by ordinary (ODE) or partial (PDE) differential equations. At first, deterministic approaches have been used to solve such problems where all parameters are considered to be deterministic. However, uncertainty plays an important role due to variations in the material, variations in the external loads, etc. Therefore, the probabilistic nature of these problems should be considered to provide a more realistic description of real-world problems.

This paper focuses on the optimal design of a loaded frame structure with a random Young's modulus. The reliability of the design is also an important issue here. Various approaches can be used to solve such a problem. Deterministic and reliability-based structural optimization of concrete cross sections has been compared, e. g. in [1], probability-based methods have been used, e.g. to optimize the spun concrete pole in [2] and heuristic algorithms have been tested, e.g. in [3]. A stochastic programming approach has been applied to a problem concerning the optimal design of beam cross section dimensions, e.g. in [4], [5], [6]. The reliability of the design has been addressed, e.g. in [7] and [8] by using chance constrained stochastic programming. This method is also utilized in this paper, and the chance constraint is handled by two approaches. Moreover, a posteriori check of satisfying the chance constraint is made, and the upper bounds of the obtainable reliability are computed.

PROBLEM FORMULATION

The problem of designing a frame structure has been chosen (see Figure 1). The task is to obtain an optimal design of rectangular cross section dimensions while the frame's weight is minimized and its rigidity is maximized. The weighted sum method is used to solve the multi-criteria problem (see [9]). A random Young's modulus related to different material characteristics is assumed. The problem can be formulated as the following ODE constrained stochastic optimization program:

$$\min_{a,b_1,b_2,b_3,v_1,v_2,v_3} \left(-\alpha\mathbb{E}\left(\frac{E(\omega)ab_1^3+E(\omega)ab_2^3+E(\omega)ab_3^3}{12c_{rigidity}}\right)+\beta\frac{\rho ab_1l_1+\rho ab_2l_2+\rho ab_3l_2}{c_{weight}}\right) \quad (1)$$

$$\text{s.t.} \quad E(\omega)\frac{ab_1^3}{12}\frac{\mathrm{d}^4v_1}{\mathrm{d}x^4}(\omega,x)=f(x), x\in\langle 0,l_1\rangle, \omega\in\Omega, \quad (2)$$

$$E(\omega)\frac{ab_2^3}{12}\frac{\mathrm{d}^4v_2}{\mathrm{d}y^4}(\omega,y)=0, y\in\langle 0,l_2\rangle, \omega\in\Omega, \quad (3)$$

$$E(\omega)\frac{ab_3^3}{12}\frac{\mathrm{d}^4v_3}{\mathrm{d}y^4}(\omega,y)=0, y\in\langle 0,l_2\rangle, \omega\in\Omega, \quad (4)$$

$$v_1(\omega,0)=0, v_1(\omega,l_1)=0, \omega\in\Omega, \quad (5)$$

$$v_2(\omega,0)=0, \frac{\mathrm{d}v_2}{\mathrm{d}y}(\omega,0)=0, v_2(\omega,l_2)=0, \omega\in\Omega, \quad (6)$$

$$v_3(\omega,0)=0, \frac{\mathrm{d}v_3}{\mathrm{d}y}(\omega,0)=0, v_3(\omega,l_2)=0, \omega\in\Omega, \quad (7)$$

$$-\frac{\mathrm{d}v_1}{\mathrm{d}x}(\omega,0)=\frac{\mathrm{d}v_2}{\mathrm{d}y}(\omega,l_2), \omega\in\Omega, \quad (8)$$

$$\frac{\mathrm{d}v_1}{\mathrm{d}x}(\omega,l_1)=-\frac{\mathrm{d}v_3}{\mathrm{d}y}(\omega,l_2), \omega\in\Omega, \quad (9)$$

$$E(\omega)\frac{ab_1^3}{12}\frac{\mathrm{d}^2v_1}{\mathrm{d}x^2}(\omega,0)=-E(\omega)\frac{ab_2^3}{12}\frac{\mathrm{d}^2v_2}{\mathrm{d}y^2}(\omega,l_2), \omega\in\Omega, \quad (10)$$

$$E(\omega)\frac{ab_1^3}{12}\frac{\mathrm{d}^2v_1}{\mathrm{d}x^2}(\omega,l_1)=E(\omega)\frac{ab_3^3}{12}\frac{\mathrm{d}^2v_3}{\mathrm{d}y^2}(\omega,l_2), \omega\in\Omega \quad (11)$$

$$\left|E(\omega)\frac{\mathrm{d}^2v_1}{\mathrm{d}x^2}(\omega,x)\frac{b_1}{2}\right|\leq\sigma_{limit}, x\in\langle 0,l_1\rangle, \omega\in\Omega, \quad (12)$$

$$\left|E(\omega)\frac{\mathrm{d}^2v_2}{\mathrm{d}y^2}(\omega,y)\frac{b_2}{2}+\frac{\int_0^{\frac{l_1}{2}}|f(x)|\mathrm{d}x}{ab_2}\right|\leq\sigma_{limit}, y\in\langle 0,l_2\rangle, \omega\in\Omega, \quad (13)$$

$$\left|E(\omega)\frac{\mathrm{d}^2v_3}{\mathrm{d}y^2}(\omega,y)\frac{b_3}{2}-\frac{\int_{\frac{l_1}{2}}^{l_1}|f(x)|\mathrm{d}x}{ab_3}\right|\leq\sigma_{limit}, y\in\langle 0,l_2\rangle, \omega\in\Omega, \quad (14)$$

$$\int_{0}^{\frac{l_1}{2}} |f(x)|\mathrm{d}x \leq \frac{\pi^2 E(\omega) a b_2^3}{12(\gamma l_2)^2}, \int_{\frac{l_1}{2}}^{l_1} |f(x)|\mathrm{d}x \leq \frac{\pi^2 E(\omega) a b_3^3}{12(\gamma l_2)^2}, \omega \in \Omega, \quad (15)$$

$$a_{min} \leq a \leq a_{max}, \quad (16)$$

$$b_{min} \leq b_1, b_2, b_3 \leq b_{max}, \quad (17)$$

where $\alpha, \beta > 0$ are the weighting coefficients, $\alpha + \beta = 1$, $\mathbb{E}(\cdot)$ denotes the expected value, $E(\omega)$ is a random Young's modulus, ω is a random outcome, Ω is a sample space, $c_{rigidity}, c_{weight}$ are typical values of rigidity and weight of the frame (normalizing constants), ρ is the beam density, l_1, l_2 are the beam lengths, x, y are the related space coordinates, $f(x)$ is a deterministic symmetric static load, a, b_1, b_2, b_3 are the first-stage decision variables (dimensions of the rectangular cross sections) and $v_1(\omega, x), v_2(\omega, y), v_3(\omega, y)$ are the second-stage decision variables (deflections). More precisely, the deflections depend on a, b_1, b_2, b_3, i.e. $v_1(\omega, a, b_1, b_2, b_3, x)$, $v_2(\omega, a, b_1, b_2, b_3, y)$, $v_3(\omega, a, b_1, b_2, b_3, y)$, but this notation is not used for the clarity of the text.

Transverse deflections of the beams in the structure are described by the ODEs (2)-(4). The boundary conditions for clamped end points given by (6) and (7) mean that there are zero transverse deflections and their slopes. Because only small deformations with respect to the dimensions of the beams are assumed, the theory of linear elasticity is used and, therefore, there are zero transverse deflections in the beam connections, see the constraints (5)-(7). The beam connections are solid, i.e. the right angles must be preserved during the deflection, see the constraints (8)-(9). Furthermore, the bending moments given as $M(x) = -EJ\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}(x)$, $M(y) = -EJ\frac{\mathrm{d}^2 v}{\mathrm{d}y^2}(y)$ ($J = \frac{ab^3}{12}$ is the second moment of the rectangular cross section) must be the same in the beam connections, see the constraints (10)-(11). The maximum value of the stress must be bounded for safety reasons. The horizontal beam is influenced only by bending moment, see the constraint (12), but the vertical beams are pressure and bending loaded, see the constraints (13)-(14). The limiting value σ_{limit} relates to the proportional limit which marks the end of the area of elastic behaviour described by Hooke's law [10]. Also, the buckling stability conditions must be added for the vertical beams, see (15), where γ is the effective length factor and γl_2 is the effective length of the vertical beams. Finally, the dimensions of the beam cross section must be bounded, see (16) and (17).

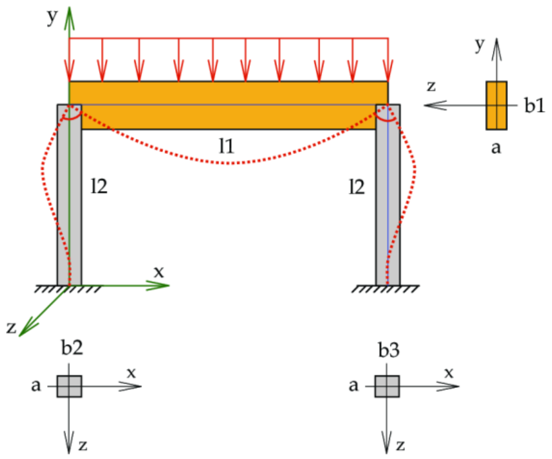

Fig. 1 – Scheme of the loaded frame structure and its cross sections with dimensions to optimize

Approximation

The approximation of the above-mentioned continuous stochastic program is made in two steps. First, a scenario-based approach for a random variable approximation is used, see e.g. [11], [12]. It is assumed that the random Young's modulus $E(\omega)$ has a discrete probability distribution with a finite number R of equiprobable scenarios $E(\omega_s)$ with the corresponding probabilities $p_s = P(\{\omega_s\}) = \frac{1}{R},\ s = 1, \ldots, R$.

The second step consists of discretizing the space coordinates x and y in the constraints. A simple finite difference method (see e.g. [13]) with a uniform grid spacing is used: $x_i = id, i = 0, \ldots, N$,
$y_j = jh, j = 0, \ldots, M, d = \frac{l_1}{N}, h = \frac{l_2}{M}$. Derivatives are replaced by the central difference formulas and difference equations are obtained. The following notation is used: $v_1(\omega, x_i) = v_{1,s,i}$, $V_{1,s,i} \approx v_{1,s,i}, \ldots$

CHANCE CONSTRAINT

A very important issue regarding the engineering design is its reliability. Therefore, the so-called individual chance constraint (or probabilistic constraint) [11] is included in the model to ensure higher reliability. In the context of this paper, the reliability of the design will mean that the constraint limiting the maximum deflection of the horizontal beam should hold with high probability:

$$P(\max_{x} v_1(\omega, x) \leq A) \geq 1 - \varepsilon, \tag{18}$$

where A is the given limiting value for the maximum deflection of the horizontal beam.

Analytical approach

For the given deterministic load $f(x) = -f_0$, the following formula for the maximum deflection of the horizontal beam can be derived:

$$v_{1,max}(\omega) = \frac{C}{E(\omega)}, C = \frac{f_0 l_1^4 (15 l_2^2 b_1^6 + 8 l_1 l_2 b_1^3 b_2^3 + 8 l_1 l_2 b_1^3 b_3^3 + 4 l_1^2 b_2^3 b_3^3)}{32 a b_1^3 (3 l_2^2 b_1^6 + 4 l_1 l_2 b_1^3 b_2^3 + 4 l_1 l_2 b_1^3 b_3^3 + 4 l_1^2 b_2^3 b_3^3)}. \tag{19}$$

It means that the maximum deflection $v_{1,max}(\omega)$ is the transformed random variable. The uniformly distributed Young's modulus is assumed, i.e. $E \sim U(E_{min}, E_{max})$, and the distribution function of the random variable $v_{1,max}(\omega)$ can be obtained:

$$F_{v_{1,max}}(z) = \begin{cases} 0, z \leq \dfrac{C}{E_{max}}, \\ \dfrac{1}{E_{max} - E_{min}}\left(E_{max} - \dfrac{C}{z}\right), \dfrac{C}{E_{max}} \leq z \leq \dfrac{C}{E_{min}}, \\ 1, z \geq \dfrac{C}{E_{min}}. \end{cases} \tag{20}$$

The left-hand side of the chance constraint (18) can be written as $P\left(\max_{x} v_1(\omega, x) \leq A\right) = F_{v_{1,max}}(A)$, and therefore, the chance constraint (18) in the analytical form is the following:

$$\frac{1}{E_{max} - E_{min}}\left(E_{max} - \frac{C}{A}\right) \geq 1 - \varepsilon. \tag{21}$$

This analytical approach can usually be used only for simple problems when it is possible to derive the corresponding distribution function. For more complex cases, another approach must be used to solve chance constrained problems.

Penalty reformulation

An appropriate penalty function ν [14] is used to reformulate the chance constrained problem (1)-(18):

$$\min_{a,b_1,b_2,b_3,v_1,v_2,v_3} \begin{aligned} &-\alpha \mathbb{E}\left(\frac{E(\omega)ab_1^3 + E(\omega)ab_2^3 + E(\omega)ab_3^3}{12c_{rigidity}}\right) + \beta \frac{\rho a b_1 l_1 + \rho a b_2 l_2 + \rho a b_3 l_2}{c_{weight}} \\ &+ M_{pen}\mathbb{E}\left(\nu\left(\max_x v_1(\omega, x) - A\right)\right), \end{aligned} \tag{22}$$

where $\nu: \mathbb{R} \to \mathbb{R}_0^+$ is a continuous nondecreasing function equal to 0 on $\mathbb{R}_0^-$ and positive otherwise, M_{pen} is a penalty coefficient. Frequently used penalty function [15] is $\nu(s) = (\max(0, s))^2$ (see Equation (23)).

Hence, the approximation of the nonlinear continuous two-stage chance constrained stochastic program is given by the following nonlinear deterministic program:

$$\min_{\substack{a,b_1,b_2,b_3,\\ \mathbf{V}_{1,s},\mathbf{V}_{2,s},\mathbf{V}_{3,s}}} \left(\begin{aligned} &-\alpha \sum_{s=1}^{R} p_s \frac{E_s a b_1^3 + E_s a b_2^3 + E_s a b_3^3}{12c_{rigidity}} + \beta \frac{\rho a b_1 l_1 + \rho a b_2 l_2 + \rho a b_3 l_2}{c_{weight}} \\ &+ M_{pen} \sum_{s=1}^{R} p_s (\max(0, \max_i V_{1,s,i} - A))^2 \end{aligned} \right) \tag{23}$$

s. t.

$$ab_1^3 E_s(\mathbb{K}\mathbf{V}_{1,s} + 2d\boldsymbol{\varphi}_{1,s}) = \mathbf{f}, s = 1, \ldots, R, \tag{24}$$

$$ab_2^3 E_s(\mathbb{K}\mathbf{V}_{2,s} + 2h\boldsymbol{\varphi}_{2,s}) = \mathbf{0}, s = 1, \ldots, R, \tag{25}$$

$$ab_3^3 E_s(\mathbb{K}\mathbf{V}_{3,s} - 2h\boldsymbol{\varphi}_{2,s}) = \mathbf{0}, s = 1, \ldots, R, \tag{26}$$

$$V_{1,s,0} = 0, V_{1,s,N} = 0, V_{2,s,0} = 0, V_{2,s,M} = 0, V_{3,s,0} = 0, V_{3,s,M} = 0, s = 1, \ldots, R, \tag{27}$$

$$\varphi_{1,s} = \varphi_{2,s}, s = 1, \ldots, R, \tag{28}$$

$$\frac{ab_1^3 E_s(2V_{1,s,1} + 2d\varphi_{1,s})}{12d^2} = -\frac{ab_2^3 E_s(2V_{2,s,M-1} + 2h\varphi_{2,s})}{12h^2}, s = 1, \ldots, R, \tag{29}$$

$$\frac{ab_1^3 E_s(2V_{1,s,N-1} + 2d\varphi_{1,s})}{12d^2} = \frac{ab_2^3 E_s(2V_{3,s,M-1} - 2h\varphi_{2,s})}{12h^2}, s = 1, \ldots, R, \tag{30}$$

$$|b_1 E_s(\mathbb{C}\mathbf{V}_{1,s} + d\boldsymbol{\varphi}_{1,s}^*)| \leq d^2 \sigma_{limit} \mathbf{g}, s = 1, \ldots, R, \tag{31}$$

$$\left|\frac{b_2 E_s}{2h^2}(\mathbb{C}\mathbf{V}_{2,s} + h\boldsymbol{\varphi}_{2,s}^*) + \frac{f_0 l_1}{2ab_2}\mathbf{1}\right| \leq \sigma_{limit}\mathbf{1}, s = 1, \ldots, R, \tag{32}$$

$$\left|\frac{b_3 E_s}{2h^2}(\mathbb{C}\mathbf{V}_{3,s} - h\boldsymbol{\varphi}_{2,s}^*) + \frac{f_0 l_1}{2ab_2}\mathbf{1}\right| \leq \sigma_{limit}\mathbf{1}, s = 1, \ldots, R, \tag{33}$$

$$\frac{f_0 l_1}{2} \leq \frac{\pi^2 E_s a b_2^3}{12(\gamma l_2)^2}, \frac{f_0 l_1}{2} \leq \frac{\pi^2 E_s a b_3^3}{12(\gamma l_2)^2}, s = 1, \ldots, R, \tag{34}$$

$$a_{min} \leq a \leq a_{max}, b_{min} \leq b_1, b_2, b_3 \leq b_{max}, \tag{35}$$

where $\mathbf{V}_{1,s} = (V_{1,s,1}, \ldots, V_{1,s,N-1})^{\mathrm{T}}$, $\mathbf{V}_{2,s} = (V_{2,s,1}, \ldots, V_{2,s,M-1})^{\mathrm{T}}$, $\mathbf{V}_{3,s} = (V_{3,s,1}, \ldots, V_{3,s,M-1})^{\mathrm{T}}$ are the approximations of $v_1(\omega, x)$, $v_2(\omega, y)$, $v_3(\omega, y)$, $\mathbb{K} = \begin{pmatrix} 7 & -4 & 1 & 0 & 0 & \ldots & 0 \\ -4 & 6 & -4 & 1 & 0 & \ldots & 0 \\ 1 & -4 & 6 & -4 & 1 & \ldots & 0 \\ & & & \vdots & & & \\ 0 & \ldots & 1 & -4 & 6 & -4 & 1 \\ 0 & \ldots & 0 & 1 & -4 & 6 & -4 \\ 0 & \ldots & 0 & 0 & 1 & -4 & 7 \end{pmatrix}$ is a square matrix of order $N-1$, $\mathbb{C} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ & & \vdots & & \\ 0 & \ldots & 1 & -2 & 1 \\ 0 & \ldots & 0 & 1 & -2 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$ is a matrix of order $(M+1) \times (M-1)$, $\mathbf{f} = (12d^4 f_1, \ldots, 12d^4 f_{N-1})^{\mathrm{T}}$, $\boldsymbol{\varphi}_{1,s} = (\varphi_{1,s}, 0, \ldots, 0, \varphi_{1,s})^{\mathrm{T}}$ is a $(N-1)$ dimensional vector, $\boldsymbol{\varphi}_{2,s} = (0,0,\ldots,0,\varphi_{2,s})^{\mathrm{T}}$ is a $(M-1)$ dimensional vector, $\boldsymbol{\varphi}_{1,s}^{*} = (\varphi_{1,s}, 0, \ldots, 0, \varphi_{1,s})^{\mathrm{T}}$, $\mathbf{g} = (1,2,2,\ldots,2,2,1)^{\mathrm{T}}$ are $(N+1)$ dimensional vectors, $\boldsymbol{\varphi}_{2,s}^{*} = (0,0,\ldots,0,\varphi_{2,s})^{\mathrm{T}}$, $\mathbf{1} = (1,\ldots,1)^{\mathrm{T}}$ are $(M+1)$ dimensional vectors.

For the analytical approach, the resulting model differs in the objective function (penalty term is removed from Equation (23)) and Equation (21) is added to model the chance constraint (18).

NUMERICAL STUDY

The results are presented for the following input data. The number of grid points are $N = 50$, $M = 50$, the number of scenarios is $R = 25, 100, 200$. The load is constant: $f(x) = -f_0$, $f_0 = 10$ Nmm^{-1}. The lengths of the steel beams are $l_1 = l_2 = 1000$ mm, their density is $\rho = 7.85 \cdot 10^{-9}$ tmm^{-3}. The stress limit is $\sigma_{limit} = 100$ MPa. The limiting values of the beam dimensions are $a_{min} = b_{min} = 10$ mm, $a_{max} = b_{max} = 100$ mm. The random Young's modulus is uniformly distributed: $E_s \sim U(200 \cdot 10^3, 230 \cdot 10^3)$ MPa. The weighting coefficients are $\alpha = 0.5$, $\beta = 0.5$ and the normalizing constants are $c_{rigidity} = 5.25 \cdot 10^{12}$ Nmm^2, $c_{weight} = 0.009$ t. The limiting value is $A = 0.9$ mm.

Both programs (using analytical approach and penalty reformulation) are implemented in GAMS with solver CONOPT and ran on a PC with Intel Core i7 CPU 860 2.8GHz and 8GB RAM.

Tab. 1 - Summary of the analytical approach results

Reliability $1-\varepsilon$	**R=25** Opt.solution [mm] a	b_1	$b_2 = b_3$	**Obj. value**	**R=100** Opt.solution [mm] a	b_1	$b_2 = b_3$	**Obj. value**	**R=200** Opt.solution [mm] a	b_1	$b_2 = b_3$	**Obj. value**
0.2	10	91.72	12.82	0.4987	10	91.72	12.86	0.4991	10	91.72	12.86	0.4991
0.4	10	92.56	12.82	0.5020	10	92.56	12.86	0.5023	10	92.56	12.86	0.5024
0.6	10	93.43	12.82	0.5054	10	93.43	12.86	0.5058	10	93.43	12.86	0.5058
0.8	10	94.33	12.82	0.5089	10	94.33	12.86	0.5093	10	94.33	12.86	0.5093
0.9	10	94.80	12.82	0.5107	10	94.80	12.86	0.5111	10	94.80	12.86	0.5111
0.99	10	95.22	12.82	0.5124	10	95.22	12.86	0.5128	10	95.22	12.86	0.5128
0.9999	10	95.27	12.82	0.5126	10	95.27	12.86	0.5129	10	95.27	12.86	0.5130
1	10	95.27	12.82	0.5126	10	95.27	12.86	0.5130	10	95.27	12.86	0.5130

Table 1 shows the summary of the analytical approach results. More precisely, the influence of the right-hand side $1-\varepsilon$ of the chance constraint (18) on the optimal dimensions for three different number of scenarios R is shown. There is a little difference only in dimensions b_2 and b_3 for different R.

The point estimate of probability (P.E. of probability.) in Table 2 is computed as the relative frequency of the event $\max_x v_1(\omega, x) \leq A$. The summary of the penalty reformulation results is presented in Table 2 shows the influence of the penalty coefficient value M_{pen} on the optimal dimensions and also on the probability of satisfying $\max_x v_1(\omega, x) \leq A$ for three different number of scenarios R. With an increasing penalty coefficient, the dimension b_1 and also the rigidity of the beam increase, and its deflection decreases. Therefore, the probability of satisfying $\max_x v_1(\omega, x) \leq A$ must also increase, as can be seen in Figure 2.

Tab. 2 - Summary of the penalty reformulation results

Penalty coeff.	**R=25**				**R=100**				**R=200**			
	Opt.solution [mm]			**P.E. of**	**Opt.solution [mm]**			**P.E. of**	**Opt.solution [mm]**			**P.E. of**
M_{pen}	a	b_1	$b_2 = b_3$	**probab.**	a	b_1	$b_2 = b_3$	**prob.**	a	b_1	$b_2 = b_3$	**probab.**
0.5	10	89.56	12.82	0.00	10	89.65	12.86	0.00	10	89.67	12.86	0.000
0.7	10	90.47	12.82	0.00	10	90.56	12.86	0.00	10	90.58	12.86	0.000
0.9	10	91.02	12.82	0.08	10	91.11	12.86	0.04	10	91.13	12.86	0.045
1	10	91.24	12.82	0.08	10	91.32	12.86	0.06	10	91.35	12.86	0.070
5	10	93.36	12.82	0.48	10	93.45	12.86	0.45	10	93.48	12.86	0.490
10	10	93.82	12.82	0.60	10	93.93	12.86	0.64	10	93.99	12.86	0.635
50	10	94.44	12.82	0.76	10	94.60	12.86	0.83	10	94.65	12.86	0.830
100	10	94.57	12.82	0.84	10	94.77	12.86	0.90	10	94.83	12.86	0.895
1000	10	94.90	12.82	0.96	10	95.09	12.86	0.97	10	95.13	12.86	0.965
10000	10	94.95	12.82	0.96	10	95.23	12.86	0.99	10	95.23	12.86	0.985
50000	10	94.96	12.82	0.96	10	95.25	12.86	0.99	10	95.25	12.86	0.990
10^7	10	94.96	12.82	0.96	10	95.25	12.86	0.99	10	95.25	12.86	0.995

The comparison of the analytical approach and the penalty reformulation is made for $R = 200$ scenarios. Several values of the penalty coefficients corresponding to the selected reliabilities are determined using the graph in Figure 3. The solutions for both approaches are quite comparable (see Table 3). Also, the computational times are similar (~ 8 min). The penalty reformulation is useful for more complex problems when the analytical approach cannot be used.

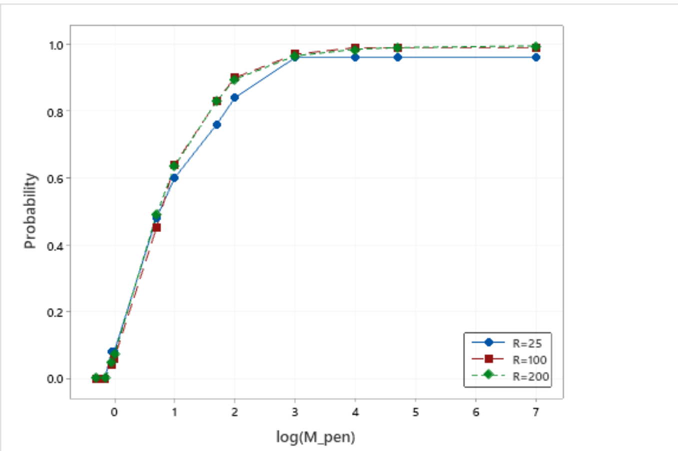

Fig. 2 – Probability of satisfying the constraint $\max_x v_1(\omega, x) \leq A$

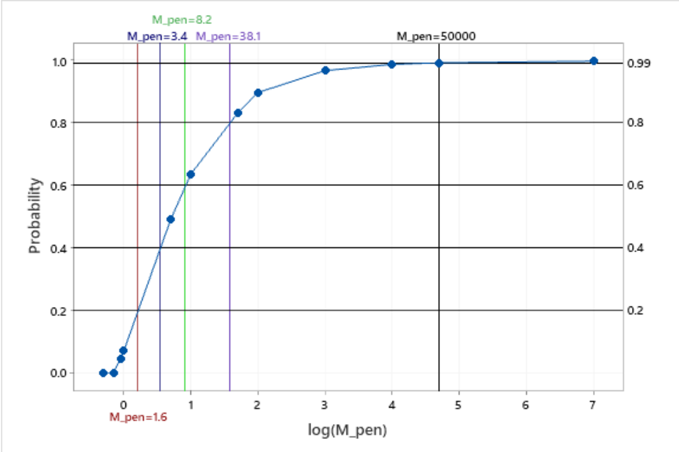

Fig. 3 – Determining the penalty coefficients for $R = 200$

Tab. 3 - Comparison of the analytical approach and penalty reformulation for $R = 200$

Reliability	**Analytical approach**				**Penalty reformulation**				***Penalty coeff.***
	Optimal solution [mm]			**Objective value**	**Optimal solution [mm]**			**Objective value**	
$1-\varepsilon$	a	b_1	$b_2 = b_3$		a	b_1	$b_2 = b_3$		M_{pen}
0.2	10	91.72	12.86	0.4991	10	92.15	12.86	0.5047	1.6
0.4	10	92.56	12.86	0.5024	10	93.12	12.86	0.5073	3.4
0.6	10	93.43	12.86	0.5058	10	93.86	12.86	0.5092	8.2
0.8	10	94.33	12.86	0.5093	10	94.56	12.86	0.5111	38.1
0.99	10	95.22	12.86	0.5128	10	95.24	12.86	0.5129	50000

RELIABILITY CHECK

A disadvantage of the penalty reformulation is that the reliability $1-\varepsilon$ from the chance constraint (18) is not involved in the model. Therefore, it would be useful to make an a posteriori check of satisfying this constraint.

First, the penalty reformulated problem with a smaller sample to yield a candidate first-stage solution $\hat{a}, \hat{b}_1, \hat{b}_2, \hat{b}_3$ is solved. Then, this solution is fixed and an a posteriori check is conducted (see [16], [17]) to see if $P\left(\max_x v_1(\omega, x) \le A\right) \ge 1-\varepsilon$. Very large Monte Carlo sample $E(\omega_1), \dots, E(\omega_{R'})$ is taken, and it is calculated how many times the maximum deflection is smaller than the given limit, i.e. $\sum_{s=1}^{R'} \mathbb{I}\,(\max_i V_{1,s,i} \le A)$. This number is denoted as r. The random variable $Y \sim Bi(R', p)$ expresses the number of experiment realizations when the random event $\max_x v_1(\omega, x) \le A$ occurs, i.e. $p = P(\max_x v_1(\omega, x) \le A)$. And now, the lower bound of the risk level ε, for which the chance constraint (18) is still satisfied with the given confidence level $1-\delta$, can be determined.

The following statistical hypotheses must be tested: null hypothesis $H_0\colon p < 1-\varepsilon$ and alternative hypothesis $H_1\colon p \ge 1-\varepsilon$. The observed value of the test statistic is $z = \frac{\frac{r}{R'}-(1-\varepsilon)}{\sqrt{\frac{(1-\varepsilon)(1-(1-\varepsilon))}{R'}}}$. The critical region is $C = \langle u_{1-\delta}, \infty)$ where $u_{1-\delta}$ is the $(1-\delta)$-quantile of the standard normal distribution. If $z \in C$, H_0 is rejected and H_1 is not rejected at the significance level δ, i.e. the solution is accepted as feasible. To obtain the lower bound of the risk level ε, the following equation must be solved:

$$\frac{\frac{r}{R'}-(1-\varepsilon)}{\sqrt{\frac{(1-\varepsilon)(1-(1-\varepsilon))}{R'}}}=u_{1-\delta}. \tag{36}$$

The lower bound of the risk level ε, for which the chance constraint is still satisfied, i.e. the solution is accepted as feasible, with confidence $1-\delta$ is $\bar{\varepsilon}$:

$$\bar{\varepsilon}=\frac{2(R'-r)+u_{1-\delta}^2+\sqrt{4ru_{1-\delta}^2+u_{1-\delta}^4-\frac{4r^2u_{1-\delta}^2}{R'}}}{2(R'+u_{1-\delta}^2)}. \tag{37}$$

The upper bound of the obtainable reliability $1-\varepsilon$ with confidence $1-\delta$ is $1-\bar{\varepsilon}$.

If R' is a very large number ($R'=50000$ is used in this case), it is impossible to solve the optimization problem directly. However, the objective function (23) is separable for fixed a, b_1, b_2, b_3:

$$\beta\frac{\rho a(b_1l_1+b_2l_2+b_3l_2)}{c_{weight}}+ \\ +\sum_{s=1}^{R}\min_{\mathbf{V}_s}\left[-\alpha p_s\frac{E_s a(b_1^3+b_2^3+b_3^3)}{12c_{rigid}}+M_{pen}p_s(\max(0,\max_i V_{1,s,i}-A))^2\right]. \tag{38}$$

It is advantageous for a multi-core processor to compute the problem in parallel because the speed-up of the computations. The upper bounds of the obtainable reliability $1-\varepsilon$ for different penalty coefficients M_{pen} are presented in Table 4 and Figure 4. The confidence level is $1-\delta=$ $=0.999$.

Tab. 4 - Summary of the reliability check

Penalty coeff.	**R=25**				**R=100**				**R=200**			
	Opt.solution [mm]			**Upp.b.**	**Opt.solution [mm]**			**Upp.b.**	**Opt.solution [mm]**			**Upp.b.**
M_{pen}	a	b_1	$b_2=b_3$	**of** $1-\varepsilon$	a	b_1	$b_2=b_3$	**of** $1-\varepsilon$	a	b_1	$b_2=b_3$	**of** $1-\varepsilon$
0.5	10	89.56	12.82	0.0000	10	89.65	12.86	0.0000	10	89.67	12.86	0.0000
0.7	10	90.47	12.80	0.0000	10	90.56	12.86	0.0000	10	90.58	12.86	0.0000
0.9	10	91.02	12.82	0.0223	10	91.11	12.86	0.0452	10	91.13	12.86	0.0429
1	10	91.24	12.82	0.0753	10	91.32	12.86	0.0958	10	91.35	12.86	0.1144
5	10	93.36	12.82	0.5746	10	93.45	12.86	0.5954	10	93.48	12.86	0.6067
10	10	93.82	12.82	0.6766	10	93.93	12.86	0.7017	10	93.99	12.86	0.7171
50	10	94.44	12.82	0.8147	10	94.60	12.86	0.8499	10	94.65	12.86	0.8720
100	10	94.57	12.82	0.8432	10	94.77	12.86	0.8871	10	94.83	12.86	0.8935
1000	10	94.90	12.82	0.8971	10	95.09	12.86	0.9565	10	95.13	12.86	0.9589
10000	10	94.95	12.82	0.8971	10	95.23	12.86	0.9888	10	95.23	12.86	0.9814
50000	10	94.96	12.82	0.8971	10	95.25	12.86	0.9935	10	95.25	12.86	0.9935
10^7	10	94.96	12.82	0.8971	10	95.25	12.86	0.9941	10	95.25	12.86	0.9941

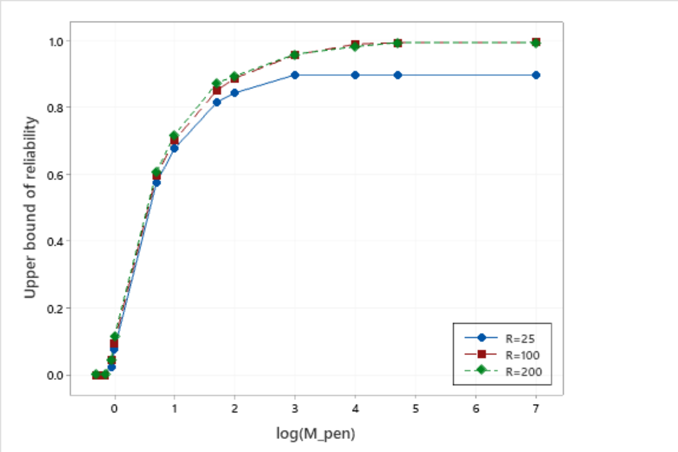

Fig. 4 – Upper bound of the obtainable reliability $1 - \varepsilon$

CONCLUSION

The applicability of the chance constrained stochastic programming approach to the civil engineering problem (a frame structure design) with a random parameter (Young's modulus) has been discussed. It has been shown that the chance constraint relates to the reliability of the structure. Two possible approaches have been used – the analytical approach and penalty reformulation. Both approaches have provided comparable solutions. Unlike the analytical approach, the penalty reformulation is applicable also for more complex problems. An a posteriori check of satisfying the chance constraint has been made, and the upper bounds of the obtainable reliability have been computed for different number of scenarios and penalty coefficients. Further research should focus on reaching higher reliability and also on problems with more random variables and joint chance constraints.

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